Algorithme pour le mouvement par courbure cristalline

Antonin Chambolle

CEREMADE, CNRS, Univ. Paris-Dauphine / PSL,
and Mokaplan, INRIA Paris

avec D. DeGennaro (CEREMADE/Parme) et M. Morini (Parme)

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Outline

- Implicit discretization of mean curvature flow: an elementary approach
- Extensions, anisotropies, crystalline case
- Discrete lattices and discrete scheme
- Consistency as $\varepsilon, h \to 0$
- Numerics

Implicit MCF for boundaries [Luckhaus-Sturzenhecker 1995], [Almgren-Taylor-Wang 1993]

E^0 given, $h > 0$ time-step, for each n E^{n+1} is found by:

$$\min_E \mathrm{Per}(E) + \frac{1}{h}\int_{E\triangle E^n} \mathrm{dist}(x, \partial E^n)dx$$

so that on ∂E^{n+1}:

$$\pm\,\mathrm{dist}(x, \partial E^n) = h\kappa_{\partial E^{n+1}}(x).$$

(Almost) equivalent formulation (cf. [C, 2004])

$$\min_u \int |Du| + \frac{1}{2h}\int (u - d_{E^n})^2 dx \quad\longrightarrow\quad E^{n+1} = \{u \le 0\}$$

where $d_{E^n} = \mathrm{dist}(x, E^n) - \mathrm{dist}(x, {}^{\complement}E^n)$ is the *signed distance* to ∂E^n. Based on this formulation, (very) simple proof of consistency in [C-Novaga, 2007].

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$$\min_E \mathrm{Per}(E) + \frac{1}{h}\int_{E\triangle E^n} \mathrm{dist}(x, \partial E^n)dx \leftrightarrow \min_E \mathcal{E}(E) + \tfrac{1}{2h}\,\mathrm{dist}(E, E^n)^2$$

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Implicit MCF for boundaries

Almost equivalent formulation: solve

$$\begin{cases} -h\,\mathrm{div}\, z^{n+1} + u^{n+1} = d_{E^n} \\ |z^{n+1}| \le 1, \quad z^{n+1} \cdot Du^{n+1} = |Du^{n+1}| \end{cases} \text{in } \mathbb{R}^N \quad \longrightarrow E^{n+1} = \{u^{n+1} \le 0\}$$

which is the Euler-Lagrange eq. for the previous problem (but makes sense also for unbounded ∂E^n).

Then (easy by comparison), d_{E^n} 1-Lipschitz $\Rightarrow$ u^{n+1} 1-Lipschitz (and $z \cdot Du = |Du|$ reads $z = \nabla u / |\nabla u|$ a.e. where $\nabla u \neq 0$, or equivalently $z \in \partial |\cdot|(\nabla u)$ a.e.)

$\longrightarrow$ follows another very simple scheme of proof of consistency:

Implicit MCF for boundaries

Since u^{n+1} is 1-Lipschitz, one has $\begin{cases} d_{E^{n+1}}(x) \geq u^{n+1}(x) & \text{where } u^{n+1} > 0 \\ d_{E^{n+1}}(x) \leq u^{n+1}(x) & \text{where } u^{n+1} < 0 \end{cases}$, so that:

$$\frac{u^{n+1} - d_{E^n}}{h} = \operatorname{div} z^{n+1}$$

implies:

$$\begin{cases} \dfrac{d_{E^{n+1}} - d_{E^n}}{h} \geq \operatorname{div} z^{n+1} & \text{where } d_{E^{n+1}} > 0 \\ \dfrac{d_{E^{n+1}} - d_{E^n}}{h} \leq \operatorname{div} z^{n+1} & \text{where } d_{E^{n+1}} < 0. \end{cases}$$

Implicit MCF for boundaries

so if $E_h(t) = E^{[t/h]}$, $z_h(t) = z^{[t/h]}$ ($[\cdot]$ =integer part), and $d_{E_h(t)}(x) \to d(x,t)$, $z_h \to z(x,t)$ as $h \to 0$, one has trivially:

$$\frac{\partial d}{\partial t} \geq \operatorname{div} z \text{ where } d > 0, \quad \frac{\partial d}{\partial t} \leq \operatorname{div} z \text{ where } d < 0.$$

at least in the distributional sense. Together with $z \in \partial|\cdot|(\nabla d)$ (i.e., $= \nabla d$ here), this is

$$\frac{\partial d}{\partial t} \geq \Delta d \text{ where } d > 0, \quad \frac{\partial d}{\partial t} \leq \Delta d \text{ where } d < 0$$

which characterizes the fact that $d(x,t) = \operatorname{dist}(x, E(t)) - \operatorname{dist}(x, {}^{\complement}E(t))$ where $E(t)$ evolves by its mean curvature (*cf* in particular [Soner, 1993]).

Implicit MCF for boundaries

Non obvious points:

- d_{E_h} is 1-Lipschitz in space but what regularity in time to get some (useful) compactness?
- $z_h \overset{*}{\rightharpoonup} z$, $z_h \in \partial|\cdot|(\nabla u_h)$, $u_h \to d$... but how can we be sure that in the limit $z \in \partial|\cdot|(\nabla d)$?

Needs a bit of control of d_{E_h} and $\operatorname{div} z_h$.

Control of d_h in time / of $\operatorname{div} z_h$

This control is obtained by understanding how a basic shape evolves, and using comparison. Isotropic case: we estimate the (time-discrete) evolution of a ball. For $B(0,R)$, $d_E(x) = |x| - R$ and one has the explicit solution:

$$\begin{cases} -h \operatorname{div} z + u = |x| - R \\ |z| \le 1, z \cdot Du = |Du| \end{cases} \longrightarrow u(x) = \begin{cases} |x| - R + h\frac{N-1}{|x|} & |x| \ge C\sqrt{h} \\ C'\sqrt{h} & |x| \le C\sqrt{h} \end{cases}$$

so that if $E' = \{u \le 0\}$, $d_{E'} \approx |x| - R + Ch/R$.

This allows to control, by comparison, both the variation of d_h and $\operatorname{div} z_h$:

- if $d_h(t,x) \ge R > 0$, then for $s > t$, $d_h(s,x) \ge R - \frac{C}{R}(s-t)$ for a short time,
- but also, we deduce $\operatorname{div} z_h \le \frac{C}{R}$ where $d_h \ge R$.

[We use $\operatorname{div} z^{n+1} = (u^{n+1} - d_{E^n})/h$ and a control from above for u^{n+1}.]

Extensions [C.-Morini-Ponsiglione 2017, C.-Morini-Novaga-Ponsiglione 2019]

- forcing term $V_n = -\kappa + g(x,t)$,
- (convex) mobility $V_n = -\psi(\nu_E)(\kappa + g)$,
- anisotropic surface tension $V_n = -\psi(\nu_E)(\kappa_\phi + g)$ where $\kappa_\phi = \mathrm{div}_\tau \nabla\phi(\nu_E)$,
- *crystalline* surface tension, case $\{\phi \le 1\}$ polytope (or simply nonsmooth, non elliptic) (see also [Giga-Giga 2001], [Giga-Pozar 2018]).

Extensions [C.-Morini-Ponsiglione 2017, C.-Morini-Novaga-Ponsiglione 2019]

• Mobility: replace Euclidean distance with ψ°-distance:
$d_E^{\psi^\circ}(x) = \min_{y\in E}\psi^\circ(x-y) - \min_{y\notin E}\psi^\circ(y-x)$, where $\psi^\circ(x) = \sup_{\psi(\nu)\le 1}\nu\cdot x$ is the polar [$\rightarrow$ distance is now measured along the vector field $\nabla\psi(\nu)$[1]];

• Surface tension: replace $\int|Du|$ replaced with $\int\phi(-Du)$[2], $\mathrm{Per}(E)$ with $\int_{\partial E}\phi(\nu)$, and the equation with:

$$-h\,\mathrm{div}\,z + u = d_E, \quad z\in\partial\phi(\nabla u) \text{ a.e.},$$

$\rightarrow$ then do exactly the same.

[1] In case $\psi = \phi$ this is called the "Cahn-Hoffmann" vector field.
[2] in the sequel we assume ϕ is even.

Extensions [C.-Morini-Ponsiglione 2017, C.-Morini-Novaga-Ponsiglione 2019]

In particular, in the simpler case $\psi = \phi$ (which we consider from now on), the explicit solution for the ball is replaced with the explicit solution for the "Wulff shape" $\{\phi^\circ \le R\}$ (which is the shape which is autosimilar for the motion):

$$\begin{cases} -h\,\mathrm{div}\, z + u = \phi^\circ(x) - R \\ z \in \partial\phi(\nabla u) \end{cases} \longrightarrow u(x) = \begin{cases} \phi^\circ(x) - R + h\frac{N-1}{\phi^\circ(x)} & \phi^\circ(x) \ge C\sqrt{h} \\ C'\sqrt{h} & \phi^\circ(x) \le C\sqrt{h} \end{cases}$$

and the proof then goes on as in the standard case ($+$ notions of distributional super/subflows with a comparison result).

Comparison / (generic) uniqueness is proved in [CMP 2017, CMNP 2019].

Fully discrete case?

In addition to $h > 0$ we consider $\varepsilon > 0$ a space discretization step. We consider a lattice ($\varepsilon\mathbb{Z}^N$ to make things simple), viewed as a graph with edges $\varepsilon\mathbb{Z}^N \times \varepsilon\mathbb{Z}^N$, and a discrete total variation

$$TV_\varepsilon(u) = \varepsilon^{N-1} \sum_{i,j \in \varepsilon\mathbb{Z}^N} \beta_{i,j} |u_i - u_j| \xrightarrow{\Gamma} \int \phi(Du) \quad \text{as } \varepsilon \to 0$$

where $\beta_{i,j} = \beta_{j,i} = \alpha_{(j-i)/\varepsilon}$ (*non-oriented, translation invariant*) and

$$\phi(p) = \sum_k \alpha_k |p \cdot k|.$$

We assume $\alpha_k \geq 0$, $\alpha_k = 0$ but for a **finite** set of indices.

Fully discrete case

We introduce the discrete gradient $D_\varepsilon : \varepsilon\mathbb{Z}^N \to \varepsilon\mathbb{Z}^N \times \varepsilon\mathbb{Z}^N$ and divergence D_ε^*:

$$(D_\varepsilon u)_{i,j} = \frac{u_j - u_i}{\varepsilon}, \qquad (D_\varepsilon^* z)_i = \sum_j \frac{z_{j,i} - z_{i,j}}{\varepsilon}$$

(the sum is on j's such that $\beta_{i,j} > 0$).

Discrete LS/ATW: what anisotropies?

Remark What anisotropies are reached by this construction?

$$\phi(p) = \sum_k \alpha_k |p \cdot k| = \sum_k \max_{t\in[-1,1]} t\alpha_k k \cdot p = \max \left\{ z \cdot p \,:\, z \in \sum_k \alpha_k [-k, k] \right\}$$

is the *support function* of the set

$$W_1 = \{\phi^\circ \le 1\} = \sum_k \alpha_k [-k, k]$$

which can only be of this form: a Minkovski sum of segments. That is, a "**zonotope**". In 2D, any symmetric polyhedron is a zonotope. In higher dimension it is more restrictive since also all facets of a zonotope are zonotopes...

Discrete LS/ATW

Minimization problem

$$\min_u TV_\varepsilon(u) + \frac{\varepsilon^N}{2h} \sum_{i\in\varepsilon\mathbb{Z}^N} (u_i - d_i^E)^2$$

Euler-Lagrange (or KKT), using $TV_\varepsilon(u) = \varepsilon^N \sup\{\langle z, D_\varepsilon u\rangle_{\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N} : |z_{i,j}| \le \beta_{i,j}\}$:

$$\begin{cases} h(D_\varepsilon^* z)_i + u_i = d_i^E & \text{for all } i \in \varepsilon\mathbb{Z}^N\,, \\ |z_{i,j}| \le \beta_{i,j}, \quad z_{i,j}(u_j - u_i) = \beta_{i,j}|u_i - u_j| & \text{where } \beta_{i,j} > 0. \end{cases}$$

Then iterate and send $\varepsilon, h \to 0$.

$\rightsquigarrow$ same proof?

Discrete LS/ATW [Braides, Gelli, Novaga 2010]

In [Braides, Gelli, Novaga, 2010], this is studied for $\phi(p) = |p_1| + |p_2|$, in dimension $N = 2$ ($\alpha_{(0,0),(0,1)} = \alpha_{(0,0),(1,0)} = 1$). The scheme is implemented in the standard way, with $E^{n+1} = \{i : u_i \leq 0\}$ and d^{n+1} the signed ℓ^∞-distance to E^{n+1}. In particular, at each step, *a rounding at scale* ε is applied.

They send then $\varepsilon, h \to 0$ and observe:

- if $\varepsilon \ll h$, then as in the continuous setting, convergence to a (crystalline) mean curvature flow;
- if $\varepsilon \gg h$, then the motion is blocked, all interfaces are pinned;
- if $\varepsilon \sim h$, then convergence to the curvature flow with a drift (and pinning of the interfaces of low curvature).

To get rid of the rounding effect, we need to forget about E and work only with u, d.

Discrete LS/ATW

If we want to reproduce the consistency proof as in the continuous setting, we need:

- a "redistancing" map $u^{n+1} \mapsto d^{n+1} = d[u^{n+1}]$ with $d^{n+1} \geq u^{n+1}$ where $u^{n+1} > 0$ and $d^{n+1} \leq u^{n+1}$ where $u^{n+1} < 0$;
- a control/estimate of the evolution starting from a Wulff shape $\{i \in \varepsilon\mathbb{Z}^N : \phi^\circ(i) \leq R\}$, or more precisely of the process

$$d = \phi^\circ - R \quad \xrightarrow{TV_\varepsilon-\text{minim.}} \quad u \quad \xrightarrow{\text{redistancing}} \quad d[u] \quad \overset{?}{\lesssim} \phi^\circ - R + \frac{Ch}{\phi^\circ};$$

- Extra: a "localized" (vectorial) $(\mathrm{z}^h)_i$ built from $(z_{i,j})_{(i,j)\in\varepsilon\mathbb{Z}^N\times\varepsilon\mathbb{Z}^N}$, since we want to consider its limit (and we need $-D_\varepsilon^* z^h \to \operatorname{div} \mathrm{z}$ if $\mathrm{z}^h \to \mathrm{z}$).

Consistent redistancing

We define a *(brute force)* redistancing operation as follows: given u 1-Lipschitz ($u_i - u_j \leq \phi^\circ(i-j) \forall i,j \in \varepsilon\mathbb{Z}^N$), we let:

$$\begin{cases} d^+[u]_i := \inf_{j: u_j < 0} u_j + \phi^\circ(j-i)\,, \\ sd^+[u]_i := \sup_{j: u_j \geq 0} d^+[u]_j - \phi^\circ(j-i) \end{cases}$$

and similarly $sd^-[u] = -sd^+[-u]$. (We also introduce various heuristically more precise interpolations, but sd^+ is the largest and sd^- the smallest). By construction, we immediately get that $sd^\pm$ are 1-Lipschitz (for ϕ°), and above u where u is positive, below where u is negative.

The fully discrete scheme with consistent redistancing

Given d^0 1-Lipschitz an initial "distance function", we define iteratively d^n, $n \geq 1$ by:

$$\begin{cases} h(D_\varepsilon^* z^{n+1}) + u_i^{n+1} = d_i^n & \text{for all } i \in \varepsilon\mathbb{Z}^N, \\ |z_{i,j}^{n+1}| \leq \beta_{i,j}, \quad z_{i,j}^{n+1}(u_j^{n+1} - u_i^{n+1}) = \beta_{i,j}|u_i^{n+1} - u_j^{n+1}| & \text{where } \beta_{i,j} > 0. \end{cases}$$

Then one shows as in the continuous case that u^{n+1} is 1-Lipschitz (using comparison $+$ invariance by integer translations), and we let $d^{n+1} = sd^+[u^{n+1}]$.
Then as before, $z_h = z^{[t/h]}$, $d_h = d^{[t/h]}$, etc. We get for free:

$$\frac{d_h(t+h) - d_h(t)}{h} \geq -D_\varepsilon^* z_h \text{ in } \varepsilon\mathbb{Z}^N \text{ where } d_h(t+h) \text{ is positive, etc.}$$

We still need: to control how d_h varies in time (control of the Wulff shape); to define a limiting z (with $-D_\varepsilon^* z_h \to \operatorname{div} z$).

Control of the Wulff shape

We need an estimate on u which solves:

$$\begin{cases} h(D^*_\varepsilon z) + u_i = \phi^\circ(i) - R & \text{for all } i \in \varepsilon\mathbb{Z}^N, \\ |z_{i,j}| \le \beta_{i,j}, \quad z_{i,j}(u_j - u_i) = \beta_{i,j}|u_i - u_j| & \text{where } \beta_{i,j} > 0. \end{cases}$$

Lemma $sd^+[u]_i \le \phi^\circ(i) - R + hC/\phi^\circ(i) + C'\varepsilon$ for some $C > 0$, $C' \ge 0$ and if $\phi^\circ(i) \ge C \max\{\varepsilon, \sqrt{h}\}$. If the weights $(\alpha_k)_k$ are **rational**, then $C' = 0$.

[***Remark:*** *the* $C'\varepsilon$ *comes from the redistancing,* u_i *is bounded by the first terms.*]

Hence: if $\varepsilon \sim h$ we get a control as in the continuous setting ($d_h(s) \ge d_h(t) - (C/R)(s-t)$ if $s > t$ and $d_h(t) \ge R > 0$). If the weights are rational, we get this control regardless the ratio ε/h.

Sending $\varepsilon, h \to 0$

Theorem If $\varepsilon \to 0$, $h \to 0$, and $\varepsilon \lesssim h$, then $d^h \to d$ which is the signed distance function of sets $E(t)$ evolving with the crystalline mean curvature flow:

$$V_n \in \phi^\circ(\nu_E)\kappa_\phi(x).$$

If the α_k are **rational**, then this holds however $\varepsilon, h \to 0$.

The proof is as in the continuous case, except one first needs to introduce

$$(\mathrm{z}^h)_i = \frac{1}{\varepsilon} \sum_{j \in \varepsilon\mathbb{Z}^N} z^h_{i,j}(t)(j-i) \in W_1$$

which is then shown to converge to a vector field $\mathrm{z}(x,t)$ with $\mathrm{z} \in \partial\phi(\nabla d)$.

Examples

- We solve the total variation on a graph using Y. Boykov and V. Kolmogorov's (2004) maxflow/mincut algorithm, together with D. Hochbaum's algorithm (2001/2013) for total variation + quadratic penalization (also [C.-Darbon, 2009/12]);
- The redistancing is slow (inf-convolution formula);
- For speedup, both operations are only done in a strip around the interface. Yet the redistancing becomes very inexact if the strip is too narrow, especially for complicated interaction patterns.

Examples

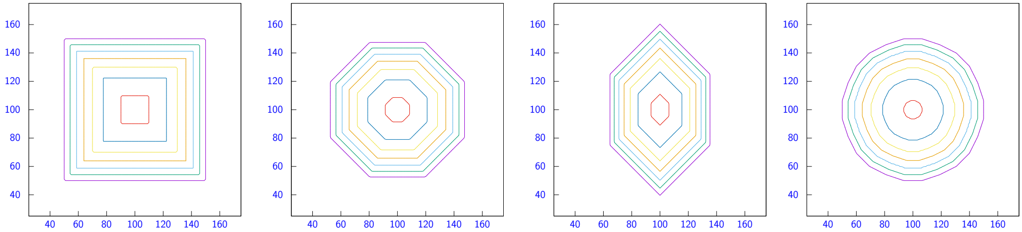

Figure: Wulff shapes of initial radius $R_0 = 50$ evolved at times $t = 0, 200, 400, \ldots, 1200$ for four different anisotropies (square, octagon, diamond and "almost isotropic").

Examples

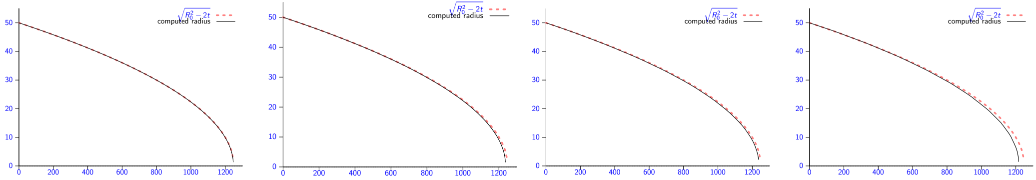

Figure: Evolution of the radius for the square, octogonal, diamond and "almost isotropic" anisotropies.

Examples

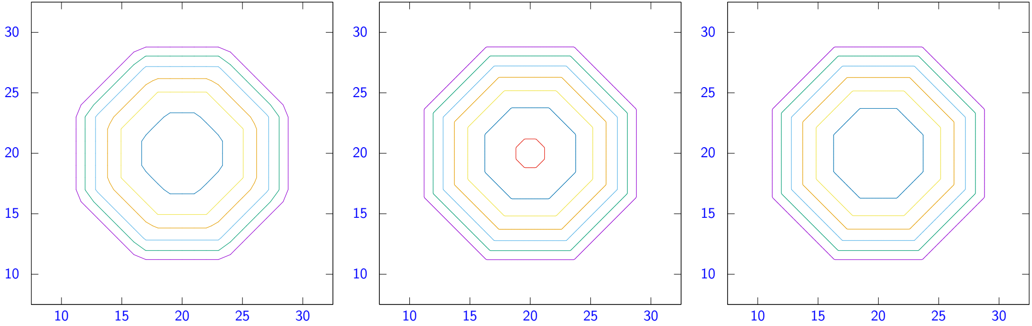

Figure: Evolution of an initial octagon with $R_0 = 10$ at times $0, 7, 14, \ldots$. Left: $\varepsilon = 1$, $h = 0.1$, middle: $\varepsilon = 0.1$, $h = 0.1$, right: $\varepsilon = 0.1$, $h = 0.5$.

Examples

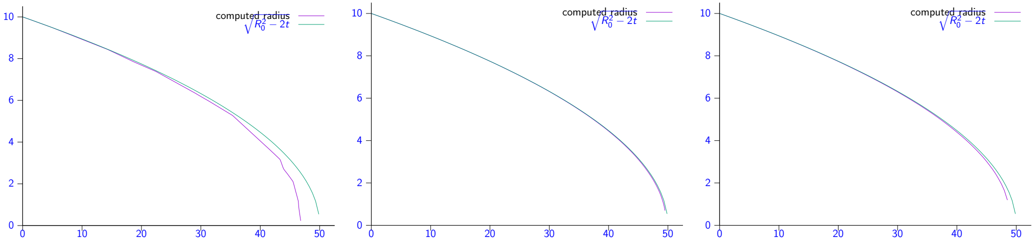

Figure: Evolution of the radius for an initial octagon with $R_0 = 10$ until the vanishing time $t = 50$. Left: $\varepsilon = 1$, $h = 0.1$, middle: $\varepsilon = 0.1$, $h = 0.1$, right: $\varepsilon = 0.1$, $h = 0.5$.

Perspectives/extensions

- *Isotropic case(!)* discretized using a $(2N+1)$-points approximation of the Laplacian and the Euclidean distance;

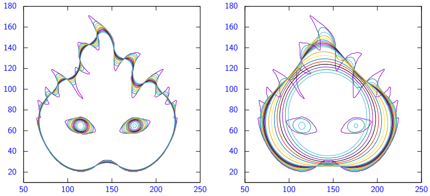

Figure: Left, $t = 0, 20, \ldots, 200$, right, $t = 0, 25, 50, \ldots, 250$ then $t = 375, 500, \ldots, 1250$.

- Nonlinear case (with a nonlinear profile $\tanh(d_E)$): partial justification of "learned" algorithms (Bretin, Denis, Masnou, Terii 2022).

Thank you for your attention