Staggered conservative scheme for the simulation of low Mach number flows

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Outline

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Low-Mach number limit problem

We are interested in the simulation of Euler barotropic equations, defined in $\Omega \subset \mathbb{R}^d \times [0, T]$:

$$\begin{cases} \partial_{\tilde{t}}\tilde{\rho} + \nabla_{\tilde{x}} \cdot (\tilde{\rho}\tilde{\mathbf{u}}) = 0 \\ \partial_{\tilde{t}}(\tilde{\rho}\tilde{\mathbf{u}}) + \nabla_{\tilde{x}} \cdot (\tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \dfrac{\nabla_{\tilde{x}}\tilde{p}}{\gamma \mathbf{M}^2} = 0 \end{cases} \tag{1}$$

with

- ρ is the density
- equation of state $p = f(\rho)$, p is the pressure
- $\mathbf{u}$ is the the velocity
- Let x_0 characteristic length, t_0 its characteristic time, ρ_0 characteristic density. $u_0 = x_0/t_0$ characteristic speed $c_0^2 = p'(\rho_0)$ characteristic sound velocity . The Mach number $\mathbf{M} := \dfrac{u_0}{c_0}$. (with $\gamma = \tilde{p}'(1)$):

$\Longrightarrow$ Singular limit

From the low Mach limit to long time limit of a wave system

A two scale asymptotic expansion, let

- $\tau = \tilde{t}/\mathbf{M}$ acoustic time scale.
- Asymptotic expansion in Mach number $\mathbf{M}$:
  $\varphi(x,t,\tau,\mathbf{M}) = \varphi(x,t,\tau)^{(0)} + \mathbf{M}\varphi(x,t,\tau)^{(1)} + \mathbf{M}^2\varphi(x,t,\tau)^{(2)} + \mathcal{O}(\mathbf{M}^3)$.

We obtain ([JP22], [Mül98]):

$$\begin{cases} \partial_\tau \tilde{\rho}^{(1)} + \nabla \cdot (\tilde{\rho}^{(0)}\tilde{\mathbf{u}}^{(0)}) = -\dfrac{d}{d\tilde{t}}\tilde{\rho}^{(0)} \\ \partial_\tau (\tilde{\rho}^{(0)}\tilde{\mathbf{u}}^{(0)}) + \tilde{c}^2(\tilde{\rho}^{(0)})\nabla\tilde{\rho}^{(1)} = 0 \end{cases} \quad (2)$$

$\rightarrow$ Low Mach number limit encapsulated in the behaviour of this wave system when

$$\tau := \frac{\tilde{t}}{\mathbf{M}} \rightarrow +\infty$$

Preservation of structures

Our problem boils down to understanding the long time limit of the wave system :

$$\begin{cases} \partial_\tau p + \dfrac{1}{\rho} div\mathbf{u} = 0 \\ \partial_\tau \mathbf{u} + \kappa \nabla p = 0 \end{cases}$$

At the continuous level :

- $\partial_\tau(\nabla^\perp \cdot \mathbf{u}) = 0$ in 2d with $\nabla^\perp \cdot \mathbf{u} = \partial_y \mathbf{u}^x - \partial_x \mathbf{u}^y$
- if boundary conditions, preservation of

$$\partial_\tau \mathbf{u}^\infty = 0, \qquad div(\mathbf{u}^\infty) = 0, \qquad \mathbf{u}^\infty \cdot \mathbf{n}_{|\partial\Omega} = \mathbf{u}_b \cdot \mathbf{n}_{|\partial\Omega}$$

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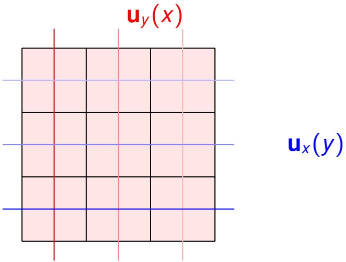
Collocated schemes

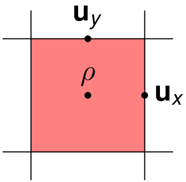
ρ, **u** at the center of the cell
Dependency on the mesh (see [DOR10] [Rie08] [Del10], [GN17])

- Godunov scheme : solving the Riemann problem at each interface introduces spurious acoustic waves even for data close to incompressible [GM04]
- Godunov scheme triangles o.k but in quads : discrete divergence free velocity space is too small ([Del10][DOR10][Rie08])



- Correction leads to centering the pressure gradient $\Longrightarrow \dfrac{\tilde{p}_{i+1} - \tilde{p}_{i-1}}{\gamma \Delta x}$

The classical staggered scheme



MAC first introduced for incompressible flows by [Har65]. Staggered scheme seems to preserve continuous structures at the discrete level such as :
de Rham complexes, sequences of the type

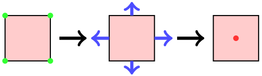
$$\{0\} \xrightarrow{id} H^1(\Omega) \xrightarrow{\nabla^\perp} H(div;\Omega) \xrightarrow{div} L^2(\Omega) \xrightarrow{0} 0$$

but **discrete**.

Staggered scheme and de Rham complexes

Discrete de Rham complex of "Nédélec-Raviart-Thomas" ([EG04],[Arn18])

$$\{0\} \xrightarrow{id} c\mathbb{Q}^1(\Omega) \xrightarrow{\nabla^\perp} \mathbb{RT}^1(\Omega) \xrightarrow{\nabla\cdot} d\mathbb{Q}^0(\Omega) \xrightarrow{0} 0$$



Important byproducts of de Rham complexes are :

- Rigourous definition of the differential operators for each space : Discrete grad, div duality, for some scalar product $(p, div\mathbf{u}) = -(\nabla_h p, \mathbf{u})$ and $(\nabla^\perp \phi, \mathbf{u}) := (\phi, (\nabla^\perp \cdot)_h \mathbf{u})$
- Hodge decomposition

$$\mathbf{u}_h = \mathbf{u}_\Psi + \mathbf{u}_\varphi \quad div(\mathbf{u}_\Psi) = 0, \quad rot(\mathbf{u}_\varphi) = 0,$$

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Starting the other way around : from the wave system to Euler

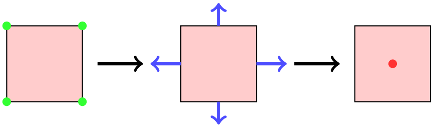
Our "algorithm" to obtain low Mach number scheme :

1) Formally, low Mach number behaviour $\approx$ long time limit of following wave system :

$$\begin{cases} \partial_\tau p + \dfrac{1}{\rho} div \mathbf{u} = 0 \\ \partial_\tau \mathbf{u} + \kappa \nabla p = 0 \end{cases}$$

2) "Good formalism" : discrete de Rham complex of "Nédélec-Raviart-Thomas"

$$\{0\} \xrightarrow{id} c\mathbb{Q}^1(\Omega) \xrightarrow{\nabla^\perp} \mathbb{RT}^1(\Omega) \xrightarrow{\nabla\cdot} d\mathbb{Q}^0(\Omega) \xrightarrow{0} 0$$



From FE to FV: the numerical scheme

Using a **mass-lumping**:

$$a(\mathbf{u}, \Psi_\sigma) = \int_\Omega \mathbf{u} \cdot \Psi_\sigma dx \text{ is replaced by } a_h(\mathbf{u}, \Psi_\sigma) := \mathbf{u}_\sigma \sum_{f \subset \mathcal{F}} \int_\Omega \Psi_f \cdot \Psi_\sigma dx.$$

Finite Volume formulation :

$$\begin{cases} |K| \partial_\tau p_K + \dfrac{1}{\rho} \displaystyle\sum_{\sigma \subset \partial K} |\sigma| \varepsilon_K(\sigma) \mathbf{u}_\sigma = \dfrac{c}{2} \displaystyle\sum_{\sigma \subset \partial K} |\sigma| [\![p]\!]_\sigma \\ |D_\sigma| \partial_\tau \mathbf{u}_\sigma + \kappa |\sigma| [\![p]\!]_\sigma = \dfrac{c}{2} |\sigma| [\![\widetilde{div} \mathbf{u}]\!]_\sigma \end{cases}$$

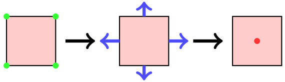
where

- $|K|$ primal volume, $|D_\sigma|$ dual volume associated to a face σ, $|\sigma|$ length of the face
- $[\![q]\!]_\sigma := q_K - q_L$, $(\widetilde{div}\mathbf{u})_K := \dfrac{1}{|\partial K|} \displaystyle\sum_{\sigma \subset \partial K} |\sigma| \varepsilon_K(\sigma) \mathbf{u}_\sigma$

Grad-div stabilization

We need stabilization because we want an explicit time integration. Recall :

$$\{0\} \xrightarrow{id} c\mathbb{Q}^1(\Omega) \xrightarrow{\nabla^\perp} \mathbb{RT}^1(\Omega) \xrightarrow{\nabla\cdot} d\mathbb{Q}^0(\Omega) \xrightarrow{0} 0$$



Since for $\Phi \in c\mathbb{Q}^1(\Omega)$ we have $\nabla^\perp \Phi \in \mathbb{RT}^1(\Omega)$: we can define

$$\langle (\nabla^\perp)^* \mathbf{u}, \Phi \rangle := \langle \mathbf{u}, \nabla^\perp \Phi \rangle$$

taking $\nabla^\perp \Phi$ as test function

$$\langle \partial_\tau (\nabla^\perp)^* \mathbf{u}, \Phi \rangle = 0 \quad \text{since} \quad \nabla(div(\nabla^\perp \Phi)) = 0$$

$\Longrightarrow$ Preservation of $(\nabla^\perp)^* \mathbf{u}$ $(\approx \nabla^\perp \cdot \mathbf{u})$.

Formally $\Delta \mathbf{u} = \nabla div(\mathbf{u}) + \nabla^\perp(\nabla^\perp \cdot \mathbf{u}) \longrightarrow$ kill $\nabla^\perp(\nabla^\perp \cdot \mathbf{u})$

Long time limit in the general case

In the general case with boundary conditions. The system preserves

$$\partial_\tau \mathbf{u}^\infty = 0, \quad div(\mathbf{u}^\infty) = 0, \quad \mathbf{u}^\infty \cdot \mathbf{n}_{|\partial\Omega} = \mathbf{u}_b \cdot \mathbf{n}_{|\partial\Omega}$$

Theorem (Hodge-Decomposition with boundary conditions)

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ an open set $\boldsymbol{u}_0, \boldsymbol{u}_b \in \mathbb{RT}^1(\Omega)$ such that $\int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} d\Gamma = 0$. Then : $\boldsymbol{u}_0 = (\boldsymbol{u}_0)_\varphi + (\boldsymbol{u}_0)_\Psi$ with

$$div(\boldsymbol{u}_0)_\Psi = 0, \quad (\boldsymbol{u}_0)_\Psi \cdot \boldsymbol{n}_{|\partial\Omega} = \boldsymbol{u}_b \cdot \boldsymbol{n}_{|\partial\Omega}$$

This comes "naturally" from the use of complexes

Theorem (Convergence in long time)

Convergence in infinity to $p^\infty = p_b$ and $\boldsymbol{u}^\infty = (\boldsymbol{u}_0)_\Psi \in \mathbb{RT}^1(\Omega)$ so $div(\boldsymbol{u}_h^\infty) = 0$, $\boldsymbol{u}^\infty \cdot \boldsymbol{n}_{|\partial\Omega} = \boldsymbol{u}_b \cdot \boldsymbol{n}_{|\partial\Omega}$

Intermediary takeways

Hodge : $(\mathbf{u}_0)_\Psi$ with $(\mathbf{u}_0)_\Psi \cdot \mathbf{n}_{|\partial\Omega} = \mathbf{u}_b \cdot \mathbf{n}_{|\partial\Omega}$

Low Mach limit $\mathbf{M} \to 0$ $\longleftrightarrow$ Long time limit $\tau := \frac{\tilde{t}}{\mathbf{M}} \to +\infty$

Complexes $\leftrightarrow$ Staggered

Grad-div stab : $\nabla(div(\nabla^\perp \cdot \mathbf{u})) = 0 \implies \partial_\tau(\nabla^\perp \cdot \mathbf{u})$

$\implies$ class of schemes Low Mach number accurate

Outline

Discretization of Euler barotropic equations in Ω

In staggered framework, mixed variables appear such as momentum : $\rho\mathbf{u}$

$\rightsquigarrow$ Convection operator is approximated in DG fashion : here $[\![\, \mathbf{q}\cdot\mathbf{n} \,]\!]_\sigma = 0$

$$\begin{cases} |K|\partial_t\rho_K + \displaystyle\sum_{\sigma\subset\partial K} |\sigma|\varepsilon_K(\sigma)\boldsymbol{q}_\sigma = \sum_{\sigma\subset\partial K} \frac{|\mathbf{u}_\sigma| + c}{2}[\![\, \rho \,]\!]_\sigma \\ |D_\sigma|\partial_t\boldsymbol{q}_\sigma + \displaystyle\sum_{K\subset\mathcal{M}} \left(-\int_K \frac{\boldsymbol{q}\otimes\boldsymbol{q}}{\rho} : \nabla\Psi_\sigma + \int_{\partial K} \boldsymbol{q}\cdot\mathbf{n}\widehat{\frac{\boldsymbol{q}}{\rho}}\cdot\Psi_\sigma d\Gamma \right) + |\sigma|[\![\, p \,]\!] = \\ \qquad \dfrac{\|\mathbf{u}\|_\infty + c}{2}|\sigma|[\![\, \widetilde{div(\boldsymbol{q})} \,]\!]_\sigma \end{cases} \tag{3}$$

with $\widehat{\frac{\boldsymbol{q}}{\rho}} = \{\{\frac{\boldsymbol{q}}{\rho}\}\}$: centered flux (no dependency with acoustic scale so only grad-div and pressure gradient appears in the asymptotic)

Conservation

The scheme is conservative in the sense that : for a fixed $\sigma \subset \mathcal{F}$

$$\int_f \boldsymbol{q} \cdot \mathbf{n}_{K,f} \frac{\widehat{\boldsymbol{q}}}{\rho} \cdot \Psi_\sigma d\Gamma = - \int_f \boldsymbol{q} \cdot \mathbf{n}_{L,f} \frac{\widehat{\boldsymbol{q}}}{\rho} \cdot \Psi_\sigma d\Gamma$$

$$\int_K \frac{\boldsymbol{q} \otimes \boldsymbol{q}}{\rho} : \nabla(\Psi_\sigma) dx = \int_K \frac{\boldsymbol{q} \otimes \boldsymbol{q}}{\rho} : \nabla(\sum_{f \neq \sigma} \alpha_f \Psi_f) dx = 0$$

for some $\alpha_f \in \mathbb{R}$

$\rightsquigarrow$ The contribution of a face $f \subset \partial K_\sigma$ is identical to all faces in ∂K_σ. This is true because

$$\text{For any } K \subset \mathcal{M}, \exists (\alpha_i)_{1 \leq i \leq 4} \text{ such that } \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sum_{f \subset \partial K} \alpha_f \Psi_f \text{ in } K$$

Outline

Numerical results: Conservativity

C++ code `Solverlab` ncells = 500

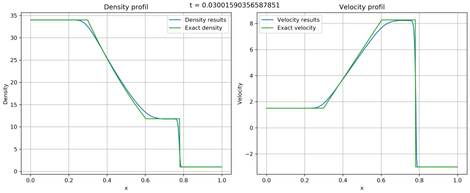


Figure 1: Explicit 1D: $p_L = 1156$, $p_R = 1$

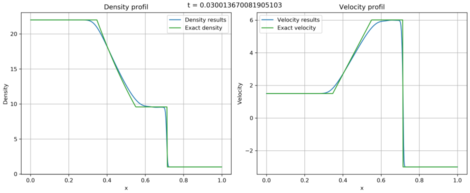


Figure 2: Implicit 1D: $p_L = 484$, $p_R = 1$

Numerical results : Cylinder Scattering I

$\mathbf{u}_0 = (0,0)^t$ $\mathbf{u}_b = (Mc(\rho_b),0)^t$ with $\rho_0 = \rho_b = 2$, Imposed boundary conditions on the outside circle (we impose $\rho_b, \mathbf{u}_b$), wall on the inside circle of the domain. $n_r = 5$, $n_\theta = 16$, $\delta t_{exp} := \frac{1}{2} \frac{min|K|}{max|\partial K|(\|\mathbf{u}^n\|_\infty + c^n))}$

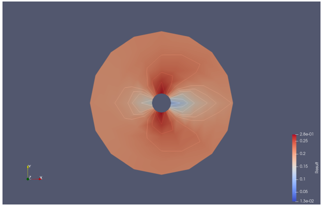


Figure 3: Explicit $M = 1e-1$

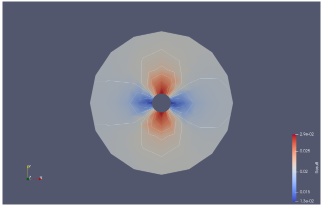


Figure 4: Explicit $M = 1e-2$

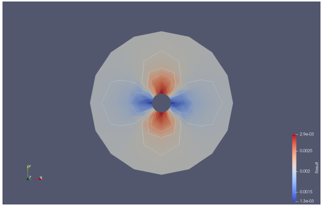


Figure 5: Explicit $M = 1e-3$

Exact solution $\mathbf{u}^\infty := Mc(\rho_b)(\mathbf{u}_0)_\Psi$

Numerical results : Cylinder Scattering II

$n_r = 10$, $n_\theta = 32$

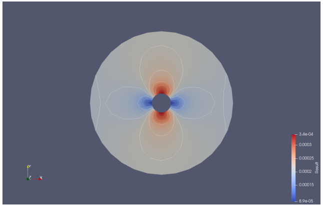


Figure 6: Semi-Implicit
$M = 1e - 4$

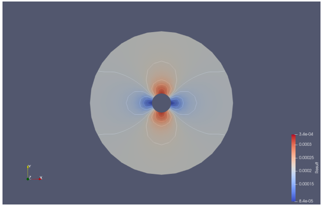


Figure 7: Exact solution
$M = 1e - 4$

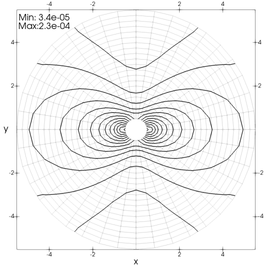


Figure 8: Roe Scheme
$M = 1e - 4$

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Conclusions and perspectives

Main takeways

i) Low Mach number behaviour : if discretization verifies

- Hodge-Helmoltz decomposition
- wave consistency
- stationary preserving diffusion

Then $\approx$ low Mach number accurate. We infer from our analysis that using staggered schemes does not imply automatically the precision at Low Mach number (non-classical grad div stabilization to get both dissipation and preservation of stationary states)

ii) Conservation

- Volume/finite Element setting to define momentum at the faces
- Conservation defined in the sense that $\sum_{f\subset\partial K} \nabla\Psi_f = 0$

What's next ? $\longrightarrow$ Extension to full Euler

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Questions?

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