The competitive spectral radius of families of nonexpansive mappings

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We introduce a new class of perfect information repeated zero-sum games (see [1]) in which the payoff of one player is the escape rate of a switched dynamical system which evolves according to a nonexpansive operator depending on the actions of both players. Formally, we fix two compact action spaces $\mathcal{A}$ and $\mathcal{B}$, a space state X equipped with a *hemi-metric* d (an assymetric metric taking possibly negative values), an initial state $x_0 \in X$ and a family of nonexpansive operators $(T_{ab})_{(a,b)\in\mathcal{A}\times\mathcal{B}}$, meaning that $d(T_{ab}(x), T_{ab}(y)) \leq d(x,y)$ for all $x, y \in X$. Two players, called *Min* and *Max*, alternatively choose actions in the sets $\mathcal{A}$ and $\mathcal{B}$. An infinite sequence of actions $(a_n, b_n)_{n\in\mathbb{N}^*}$ determines the following *escape rate* :

$$\limsup_{k\to\infty} \frac{d(T_{a_k b_k} \circ \cdots \circ T_{a_1 b_1}(x_0), x_0)}{k} .$$

Min wants to minimize it, while *Max* wants to maximize it. Taking X as the interior of a closed convex cone in $\mathbb{R}^n$ equipped with the Funk metric, we obtain subclasses of *Matrix Multiplication Games*, a 2-player extension of the joint spectral radius, introduced by Asarin et al. in [2].

Under some continuity assumptions, we prove that escape rate games have a *uniform value* ρ, which is characterized by a two-player generalization of Mañe's lemma :

$$\rho = \max_{v} \inf_{x\in X} (Sv(x) - v(x))$$

where the maximum is taken over the set of Lipschitz functions v of constant 1, and S is the self-map of the space of Lipschitz functions given by $Sv(x) := \inf_{a\in\mathcal{A}} \sup_{b\in\mathcal{B}} v(T_{ab}(x))$. This extends the maximin characterization of [3] and allows us to show the existence of optimal strategies for both players. Moreover, having S act on the set of *distance-like* functions D (f is *distance-like* if $\exists(\alpha, x^*) \in \mathbb{R}\times X$ such that $f(\cdot) \geq \alpha + d(\cdot, x^*)$), we provide a dual minimax characterization of the value

$$\rho = \inf_{v\in D} \sup_{x\in X} (Sv(x) - v(x))$$

when d is the Funk metric, or when $(T_{ab})_{(a,b)\in\mathcal{A}\times\mathcal{B}}$ are *almost isometries.*

Finally, we discuss an ongoing work with Ian Morris on the continuity and approximability of the competitive spectral radius, showing in particular that it can be approximated up to any accuracy for matrix multiplication games with sets of positive matrices.

[1] M. Akian, S. Gaubert, L. Marchesini. *The competitive spectral radius of families of nonexpansive mappings*, 2024. ArXiv :2410.21097.

[2] E. Asarin, J. Cervelle, A. Degorre, C. Dima, F. Horn, V. Kozyakin. *Entropy games and matrix multiplication games.* In *Proceedings of the 33rd International Symposium on Theoretical Aspects of Computer Science (STACS)*, vol. 47 of *LIPIcs. Leibniz Int. Proc. Inform.*, pp. 11 :1–11 :14, 2016.

[3] S. Gaubert, G. Vigeral. *A maximin characterisation of the escape rate of non-expansive mappings in metrically convex spaces.* Math. Proc. Camb. Philos. Soc., **152(2)**, 341–363, 2012.

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