Numerical optimization for off-the-grid curve reconstruction in 2D images

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UNIVERSITÉ CÔTE D'AZUR | ÉCOLE UNIVERSITAIRE DE RECHERCHE SYSTÈMES NUMÉRIQUES POUR L'HUMAIN

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MORPHEME MORPHOLOGIE ET IMAGES

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Outline

1. Introduction and context
2. The space of divergence-measure fields
3. Optimization functional
4. Numerical implementation
5. Results and perspectives

Why do we care about curves?

✤ Remote sensing
(NASA Earth Observatory, R. Simmon)

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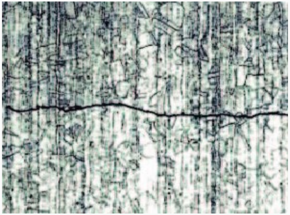

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(Institut Langevin / EDF)

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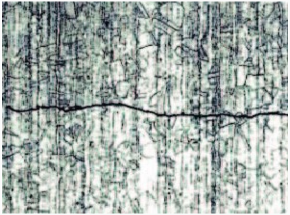

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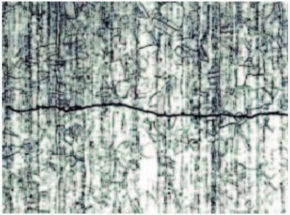

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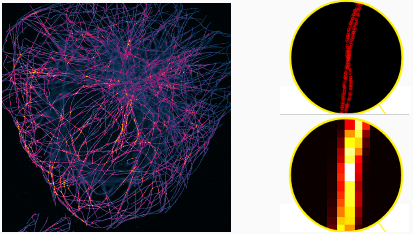

✤ Biological imaging

How do we model curve?

✤ Parametrised curve

$\gamma : [0,1] \to \mathcal{X} \subset \mathbb{R}^2$, Lipschitz map.

✤ Curve support

$\Gamma := \{\gamma(t) \mid t \in [0,1]\}$

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Scalar measure:

$$\boxed{\mathcal{M}(\mathcal{X}) = \mathcal{C}_0(\mathcal{X}, \mathbb{R})^*}$$

$$\overset{x}{\bullet} \longrightarrow \delta_x \in \mathcal{M}(\mathcal{X})$$

$\forall \phi \in \mathcal{C}_0(\mathcal{X}, \mathbb{R})$,

$$\langle \delta_x, \phi \rangle_{\mathcal{M}(\mathcal{X}) \times \mathcal{C}_0(\mathcal{X})} = \phi(x).$$

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✤ Vector measure:

$$\boxed{\mathcal{M}(\mathcal{X})^2 = \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)^*}$$

$$\mu_\gamma \in \mathcal{M}(\mathcal{X})^2$$

$\forall g \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)$,

$$\langle \mu_\gamma, g \rangle_{\mathcal{M}(\mathcal{X})^2 \times \mathcal{C}_0(\mathcal{X})^2} = \int_0^1 g(\gamma(t)) \cdot \dot{\gamma}(t) \mathrm{d}t.$$

The space of divergence measure fields $\mathcal{V}$

The space $\mathcal{V}$

$$\mathcal{V} = \left\{ m \in \mathcal{M}(\mathcal{X})^2,\ \mathrm{div}(m) \in \mathcal{M}(\mathcal{X}) \right\}$$

$\mathcal{V}$- norm : $\|\cdot\|_{\mathcal{V}}$

$$\|m\|_{\mathcal{V}} \overset{def.}{=} \|m\|_{\mathrm{TV}^2} + \|\mathrm{div}(m)\|_{\mathrm{TV}}$$

TV-norm: $\forall m \in \mathcal{M}(\mathcal{X})^2,\ \|m\|_{\mathrm{TV}^2} = \sup \left\{ \langle m, f \rangle,\ f \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2),\ \|f\|_{\infty,\mathcal{X}} \leq 1 \right\}$

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✤ **Smirnov decomposition into curves** [Smirnov, 1993][Rodríguez, 2024][Irving, 2025]

$$\mathcal{K}^{\mathrm{Smirv}} = \left\{ \gamma : [0,1] \to \mathbb{R}^2 \,\middle|\, \gamma \text{ is Lipschitz}, |\dot{\gamma}(t)| \leq 1 \text{ a.e.} \right\}$$

$\forall m \in \mathcal{V}(\mathcal{X}), \exists\, \nu \in \mathcal{M}^+(\mathcal{K}^{\mathrm{Smirv}}),$

$$m = \int_{\mathcal{K}^{\mathrm{Smirv}}} \mu_\gamma \,\mathrm{d}\nu(\gamma), \quad |m| = \int_{\mathcal{K}^{\mathrm{Smirv}}} |\mu_\gamma| \,\mathrm{d}\nu(\gamma)$$

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✤ **Extremal points** [Laville et al., 2023]

$$\mathrm{Ext}(\mathcal{B}_{\mathcal{V}}) = \left\{\frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathcal{V}}}, \gamma \text{ is a } 1\text{−rectifiable simple oriented Lipschitz curve}\right\}$$

Optimization functional (1)

Optimization functional

$$\underset{m\in\mathcal{V}(\mathcal{X})}{\arg\min}\ \underbrace{\frac{1}{2}\|y-F(m)\|^2_{L^2(\mathcal{X})}}_{\textbf{data fidelity term}} + \underbrace{\lambda\|m\|_{\mathcal{V}}}_{\textbf{regularization}}$$

- y: Image with blurred curves
  + Additive Gaussian noise
- $F:\mathcal{V}\to L^2(\mathcal{X}):m\mapsto F(m)$?

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blurred curve: γ simple parametrisation of Γ

$$(\delta_\Gamma * h)(x) = \int_0^1 h(\gamma(t) - x) \, \|\dot{\gamma}(t)\| \, \mathrm{d}t, \; \forall x \in \mathcal{X}$$

Optimization functional (2)

- **How to define** $F(m)$?

Idea

For $m \in \mathcal{V}(\mathcal{X})$,

1. Given a decomposition: $m = \int_{\mathcal{K}} \mu_\gamma \, \mathrm{d}\nu(\gamma)$

2. $$F(m)(x) = \int_{\mathcal{K}} (\delta_{\gamma([0,1])} * h)(x) \, \mathrm{d}\nu(\gamma)$$

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Example:

For $m = \sum_{i=1}^{N} a_i \, \mu_{\gamma_i}$; $F(m)(x) = \sum_{i=1}^{N} a_i \left(\delta_{\gamma_i([0,1])} * h\right)(x)$

$$= \sum_{i=1}^{N} a_i \int_0^1 h(\gamma_i(t) - x) \, \|\dot{\gamma}(t)\| \, \mathrm{d}t$$

$\rightsquigarrow$ if γ_i is a simple curve for all i

✤ **Can we choose $\mathcal{K}$ so that almost all curves in the decomposition are simple?**

Optimization functional (3)

✤ The space of curves $\mathcal{K}$

$$\mathcal{K} = \left\{ \gamma \in C^3([0,1];\mathbb{R}^2) \;\middle|\; \|\dot{\gamma}\| \leq 1,\ \|\ddot{\gamma}\| \leq 2\sqrt{2},\ \|\gamma^{(3)}\| \leq L,\ \|\gamma^{(4)}(t)\| \leq M \text{ a.e.} \right\}$$

✤ New Smirnov-based decomposition theorem

For all $m \in \mathcal{V}(\mathcal{X})$, $\exists\, \nu \in \mathcal{M}^+(\mathcal{K})$ such that,

$$m = \int_{\mathcal{K}} \mu_\gamma \,\mathrm{d}\nu(\gamma), \quad |m| = \int_{\mathcal{K}} |\mu_\gamma| \,\mathrm{d}\nu(\gamma), \qquad \nu\text{-a.e } \gamma \in \mathcal{K} \text{ is simple}$$

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Acquisition model F

For $m = \int_{\mathcal{K}} \mu_\gamma \, d\nu(\gamma)$,

$$F(m) : \mathcal{X} \longrightarrow \mathbb{R}$$

$$x \mapsto \int_{\mathcal{K}} \left(\int_0^1 h(\gamma(t) - x)\|\dot{\gamma}(t)\| \, dt \right) d\nu(\gamma) = \int_{\mathcal{K}} (|\mu_\gamma| * h)(x) d\nu(\gamma)$$

$$= (|m| * h)(x)$$

Optimization Functional (4)

Proposition: (Weak*-continuity of F)

F is continuous from the weak* topology of $\mathcal{V}(\mathcal{X})$ to the strong topology of $L^2(\mathcal{X})$

Proof: Let $(m_n)_n \in \mathcal{V}(\mathcal{X}),\ m \in \mathcal{V}(\mathcal{X}),$ such that $m_n \rightharpoonup^* m \in \mathcal{V}(\mathcal{X})$

❶ $\exists\, \nu_n \in \mathcal{M}^+(\mathcal{K}), \quad m_n = \int_{\mathcal{K}} \mu_\gamma d\nu_n(\gamma), \quad \nu_n(\mathcal{K}) \leq 2\|m_n\|_{\mathcal{V}}$

❷ ν_n bounded in $\mathcal{M}^+(\mathcal{K}) \implies \exists\, \nu \in \mathcal{M}^+(\mathcal{K}), \quad \nu_n \rightharpoonup^* \nu \in \mathcal{M}^+(\mathcal{K})$ (up to a subsequence)

❸ $m_n \rightharpoonup^* \int_{\mathcal{K}} \mu_\gamma \, d\nu(\gamma) \implies m = \int_{\mathcal{K}} \mu_\gamma \, d\nu(\gamma)$ by uniqueness of the weak*-limit

❹ For all $x \in \mathcal{X}$,

The map $\gamma \mapsto \int_0^1 h(\gamma(t) - x)\|\dot{\gamma}(t)\| \, dt \in C_0(\mathcal{K}) \implies F(m_n)(x) \longrightarrow F(m)(x)$

$$\implies F(m_n) \xrightarrow[n \to +\infty]{L^2(\mathcal{X})} F(m)(\text{ by Dom. Conv. Thm.})$$

Optimization Functional (5)- Existence of Solution

Recall:

$$\arg\min_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m) \overset{def.}{=} \frac{1}{2} \|y - F(m)\|^2_{L^2(\mathcal{X})} + \lambda \|m\|_{\mathcal{V}}$$

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❶ Minimizing sequence:

$$m_n = \int_{\mathcal{K}} \mu_\gamma \, d\nu_n(\gamma), \quad \nu_n(\mathcal{K}) \le 2\|m_n\|_{\mathcal{V}}, \quad T_\lambda(m_n) \longrightarrow \inf_{m\in\mathcal{V}(\mathcal{X})} T_\lambda(m).$$

❷ Coercivity of T_λ:

$$(m_n) \text{ bounded in } \mathcal{V}(\mathcal{X}) \implies \nu_n \text{ bounded in } \mathcal{M}^+(\mathcal{K}).$$

❸ Weak*-compactness of $\mathcal{M}^+(\mathcal{K})$:

$$\exists \nu \in \mathcal{M}^+(\mathcal{K}), \quad \nu_n \rightharpoonup^* \nu \in \mathcal{M}^+(\mathcal{K}), \quad m_n \rightharpoonup^* m = \int_{\mathcal{K}} \mu_\gamma \, d\nu(\gamma), \quad div(m_n) \rightharpoonup^* div(m)$$

❹ (a) $\liminf \|m_n\|_{\mathrm{TV}^2} \ge \|m\|_{\mathrm{TV}^2}$ (b) $\liminf \|\operatorname{div}(m_n)\|_{\mathrm{TV}} \ge \|\operatorname{div}(m)\|_{\mathrm{TV}}$

(c) $\lim_{n\to\infty} \|y-F(m_n)\|^2_{L^2} = \|y-F(m)\|^2_{L^2}$

$$\inf_{m\in\mathcal{V}(\mathcal{X})} T_\lambda(m) = \liminf_{n\to\infty} T_\lambda(m_n) \ge T_\lambda(m) \implies m \text{ is a minimiser of } T_\lambda$$

Numerical implementation(1)

Definition:

$$\mathcal{W} = \left\{ f\,dx \,|\, f \in L^1(\mathcal{X}, \mathbb{R}^2),\ \mathrm{div}(f) \in L^1(\mathcal{X}, \mathbb{R}) \right\}$$

Lemma: $\mathcal{W}$ is a dense subset of $\mathcal{V}$ for the weak-* topology.

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Lemma: $\mathcal{W}$ is a dense subset of $\mathcal{V}$ for the weak-* topology.

Optimization functional in $\mathcal{W}$

$$\underset{f dx\, \in \mathcal{W}(\mathcal{X})}{\arg\min}\ \frac{1}{2} \|y - |f| * h\|^2_{L^2(\mathcal{X})} + \lambda \left(\|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)} + \|\mathrm{div}(f))\|_{L^1(\mathcal{X}, \mathbb{R})} \right)$$

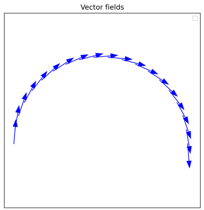
$$F(f\, dx) = |f\, dx| * h = |f| * h$$

Numerical implementation (2)

✤ Off-the-grid curve recovery

γ integral curve of f

$$\begin{cases} \dot{\gamma}(t) = f(\gamma(t)) \\ \gamma(0) = \gamma_0 \end{cases}$$



$\rightsquigarrow$ The equation is solved for multiple initializations carefully chosen to recover all the curves.

Numerical implementation(3)

✤ Discretization

- $\mathcal{X} = \bigcup_{i=1}^{N} P_i$, where $(P_i)_{1\leq i\leq N}$ forms a regular grid of $N = N_x \times N_y$ pixels.
- f is piecewise constant: $f(x) = \sum_{i=1}^{N} f_i \, \mathbb{1}_{P_i}(x), \; f_i \in \mathbb{R}^2$
- $\mathrm{div}(f)$ is discretized using finite differences
- Smooth relaxation: $|x| \approx \sqrt{x^2 + \epsilon} - \sqrt{\epsilon}$

$\rightsquigarrow$ The resulting functional is smooth and finite-dimensional

✤ Implementation

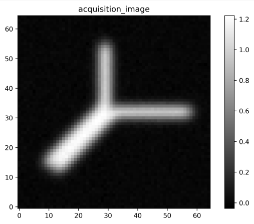
- **Optimization of the discretized functional**

  $\longrightarrow$ Solved using **L-BFGS** (quasi-Newton method)

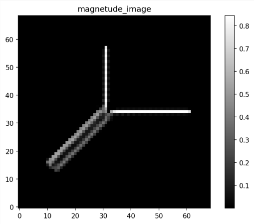
- **Numerical integration of curves**

  $\longrightarrow$ Performed via **Runge–Kutta 4**

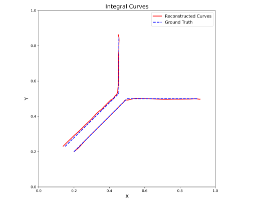
Numerical Results



a) Blurry and noisy image



b) Vector fields



c) Integral curves

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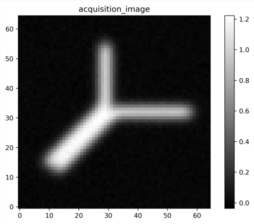

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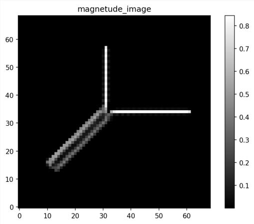

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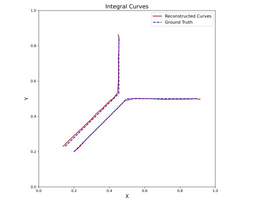

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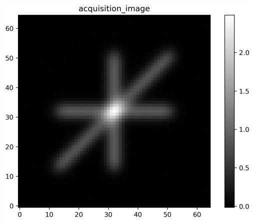
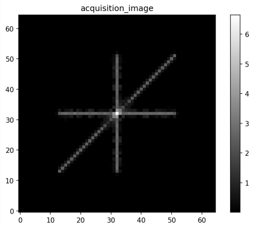
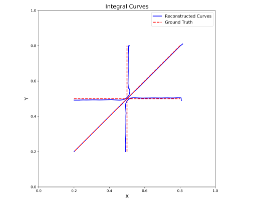


d) Blurry and noisy image



e) Vector fields



f) Integral curves

https://gitlab.inria.fr/atsafack/implementation-of-croc_relaxed-functional-on-w

Perspectives

Perspectives

- **Theoretical analysis:** Characterization of the minimizers (e.g., proof of a representation theorem)
- **Numerical study:** Deeply investigate simulation - on real data
- **Optimization:** Explore direct minimization (in $\mathcal{V}$ or $\mathcal{W}$ spaces)

References

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