Stabilization of chemostats with mortality and substrate dynamics

Epiphane Loko

Ecole nationale des ponts et chaussées, CERMICS-Univ Paris Saclay, L2S

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Supervisors: **Antoine CHAILLET and Amaury HAYAT**

Joint work with Iasson Karafyllis, Miroslav Krstic, Antoine Chaillet

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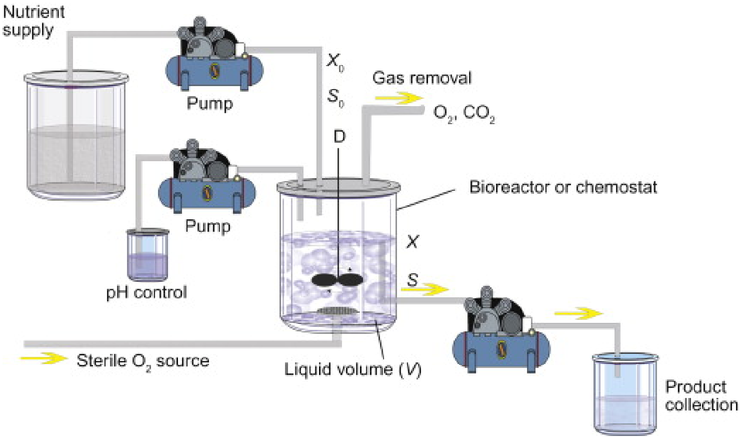
PART I

The simplest chemostat model

The chemostat: A microbial bioreactor



The chemostats are used to

1. produce antibiotics and some proteins
2. treat water (degradation of pollutant)
3. produce biomass
4. . . .

The simplest chemostat is modeled by:

$$\begin{cases} \dot{X} = (p_0\mu(S) - b - D)X \\ \dot{S} = D(S_{\text{in}} - S) - \mu(S)X. \end{cases} \tag{1}$$

Symbols	Amounts
X	Microbial concentration or biomass
S	Concentration of the substrate (nutrient)
S_{in}	Inlet concentration of the nutrient
$\mu(S)$	Specific growth rate
D	Dilution rate
b	Mortality rate
$p_0 = 1/k$	Yield factor

The parameters X, S, D, b, p_0 are positive and $\mu \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ with $\mu(0) = 0$.

The goal in the chemostat is to maintain the system in a non washout steady state.

Equilibrium point

An equilibrium point of the chemostat model (1) is (X^*, S^*) such that

$$p_0\mu(S^*) = b + D^*,$$
$$X^* = \frac{D^*(S_{\text{in}} - S^*)}{\mu(S^*)}.$$

Given $D^* > 0$, it may exist many equilibrium points

Stability

Given $D^* > 0$ and an equilibrium (X^*, S^*), the linearization of the system (1) is

$$\begin{pmatrix} \dot{X} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} 0 & p_0 \mu'(S^*)X^* \\ -(b+D^*)/p_0 & -D^* - \mu'(S^*)X^* \end{pmatrix} \begin{pmatrix} X \\ S \end{pmatrix}. \tag{2}$$

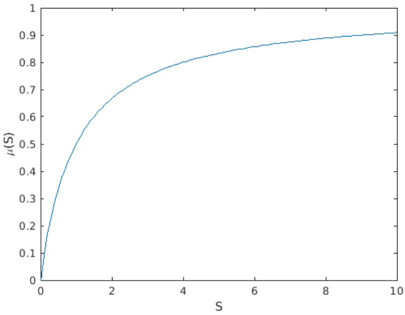
The characteristic polynomial is:

$$P(s) = s^2 + (D^* + \mu'(S^*)X^*)s + (b + D^*)\mu'(S^*)X^*. \tag{3}$$

The system is unstable if $\mu'(S^*) < 0$.

When considering the Monod kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S}, \tag{4}$$

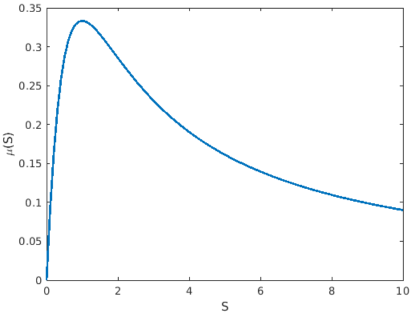


Theorem 1 (Dali-Youcef, Rapaport, and Sari 2022)

There exists a unique positive equilibrium point of (1) *which is globally asymptotically stable and exponentially stable.*

When, considering the Haldane kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S + aS^2}, \tag{5}$$



Theorem 2

The system (1) *admits two positive equilibrium points* (X^*, S^*), (X^{**}, S^{**}) *such that*

- (X^*, S^*) *is unstable.*
- (X^{**}, S^{**}) *is exponentially stable.*

Unstable open-loop chemostat model

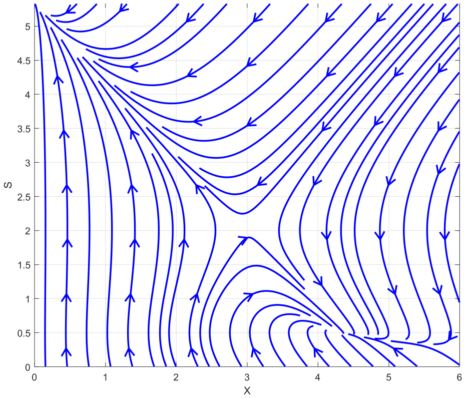


Figure: The phase diagram for the open-loop chemostat model (1) with Haldane equation

Theorem 3

Assume that

$$p_0\mu(S) > b, \quad \forall S \in [S^*, S_{in}]. \tag{6}$$

Then for every $\delta > 0$ *and* $\alpha \in [0,1)$ *the feedback law defined as*

$$D(X,S) = \frac{D^*\mu(S)X}{\mu(S^*)X^*} + \frac{\delta b}{(\mu(S^*))^{1+\alpha}} \begin{cases} |\mu(S) - \mu(S^*)|^{1+\alpha}, & \textit{if} \quad S \leq S^* \\ 0, & \textit{if} \quad S > S^* \end{cases} \tag{7}$$

achieves global asymptotic stabilization of (X^*, S^*)*. Moreover, if* $\alpha > 0$ *then the feedback law* (7) *also achieves local exponential stabilization of* (X^*, S^*)*.*

Stable closed-loop model

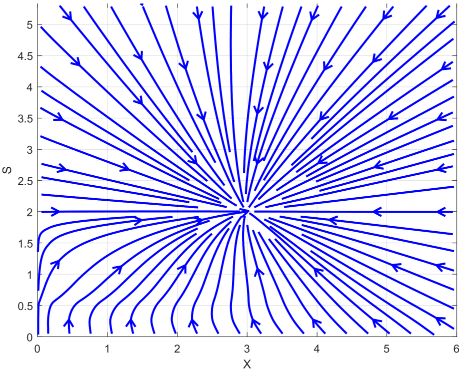


Figure: The phase diagram for the closed-loop chemostat model (1) with (7).

Is the sufficient condition (6) also necessary to ensure GAS of system (1)?

Theorem 4

Assume that there exists a constant $\overline{S} \in (S^*, S_{in})$ *such that*

$$p_0\mu(\overline{S}) < b \tag{8}$$

$$\mu'(S) \leq 0, \quad \forall S \in [\overline{S}, S_{in}]. \tag{9}$$

Then, there is no feedback law $D(X,S) \geq 0$ *that achieves global stabilization of the equilibrium* (X^*, S^*) *of* (1).

PART II

An age structured chemostat model with substrate dynamics

Consider the following age structured chemostat model with substrate dynamics

$$\frac{\partial f}{\partial t}(t,a) + \frac{\partial f}{\partial a}(t,a) = -(\beta(a) + D(t))f(t,a), \tag{10}$$

$$f(t,0) = \mu(S(t)) \int_0^{+\infty} k(a)f(t,a)da, \tag{11}$$

$$\dot{S}(t) = D(t)(S_{\text{in}} - S(t)) - \mu(S(t)) \int_0^{+\infty} q(a)f(t,a)da, \tag{12}$$

where $f(t,a) > 0$ is the distribution function of the microbial population in the chemostat at time $t \geq 0$ and age $a \geq 0$.

We derived the following ODE chemostat model

$$\dot{S} = D(S_{\mathsf{in}} - S) - \mu(S)X \tag{13a}$$

$$\dot{X} = q_0\mu(S)Y - (b+D)X \tag{13b}$$

$$\dot{Y} = p_0\mu(S)Y + \gamma X - (b+D)Y, \tag{13c}$$

where

$$X(t) = \int_0^{+\infty} q(a)f(t,a)da, \quad Y(t) = \int_0^{+\infty} k(a)f(t,a)da. \tag{14}$$

Unstable open-loop equilibrium

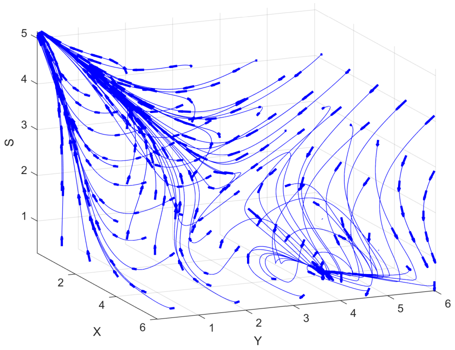


Figure: The phase diagram for the chemostat model (13) with Haldane kinetics.

(A) There exists a constant $\varphi > 1$ such that for $S \in [S^*, S_{\text{in}}]$:

$$\left(b - \frac{\lambda(b+D^*)}{2}\right)\left(\frac{\mu(S^*)}{\mu(S)} - 1\right) + (1+\lambda)\frac{\lambda^2\varphi(b+D^*)^2}{4D^*}\left(\frac{\mu(S^*)}{\mu(S)} - 1\right)^2 < D^*\left(1 - \frac{1}{4(1+\lambda)\varphi}\right)$$

Theorem 5

*Under assumption **(A)** the feedback law*

$$D(X,S) = \frac{D^*\mu(S)X}{\mu(S^*)X^*} + \frac{\delta}{(\mu(S^*))^{1+\alpha}}\begin{cases}|\mu(S) - \mu(S^*)|^{1+\alpha}, & \textit{if} \quad S \leq S^* \\ 0, & \textit{if} \quad S > S^* \end{cases} \tag{15}$$

achieves global asymptotic stabilization of the equilibrium. Moreover, if $\alpha > 0$, then the feedback law (15) *achieves in addition local exponential stabilization of the equilibrium.*

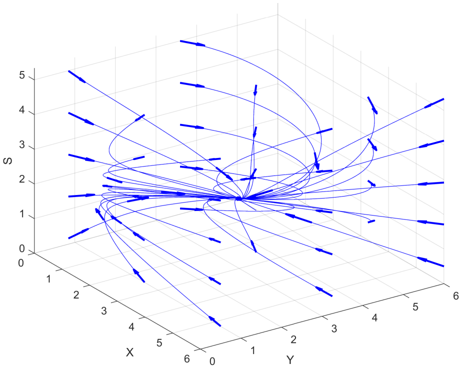


Figure: The phase diagram of the closed-loop system (13) with (15)

Perspectives

1. Consider S_{in} and/or the volume V to be depending on t
2. Consider another reproduction rate μ than Monod and Haldane
3. Investigate the stabilization of the PDE-ODE model (10)-(11)-(12)

Thank you for your attention