Stochastic homogenization of the double porosity model: when agregates are allowed

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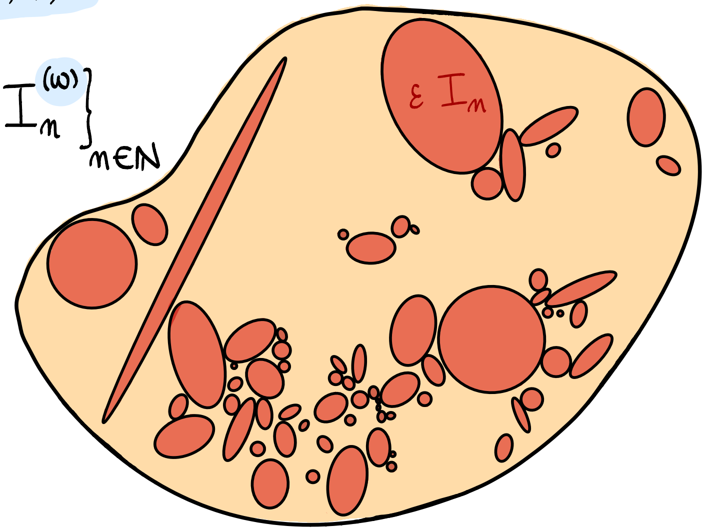
Double porosity model

$\omega \in \Omega \quad (\Omega, \mathcal{A}, \mathbb{P})$

$\{I_n^{(\omega)}\}_{n \in \mathbb{N}}$



$D \subset \mathbb{R}^d$

$$u_\varepsilon - \mathrm{div}\left(\left(1 - \chi_\varepsilon + \varepsilon^2 \chi_\varepsilon\right) \nabla u_\varepsilon\right) = f$$

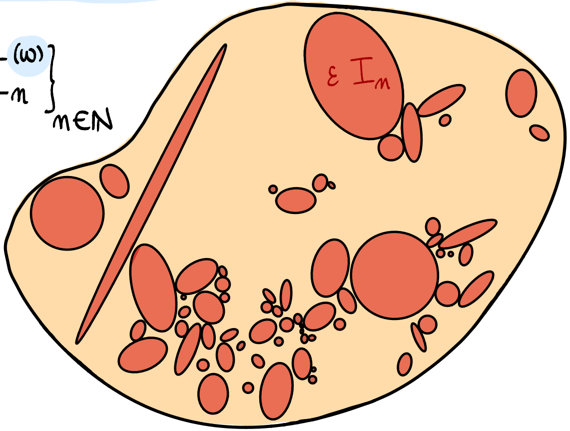
$$\chi_\varepsilon := \chi_\varepsilon^{(\omega)} = \mathbb{1}_{\bigcup_n \varepsilon I_n^{(\omega)}}$$

$$u_\varepsilon := u_\varepsilon^{(\omega)} \in H_0^1(D).$$

Asymptotic analysis: a priori estimates

$\omega \in \Omega \quad (\Omega, \mathcal{A}, \mathbb{P})$

$\{I_n^{(\omega)}\}_{n\in\mathbb{N}}$



$D \subset \mathbb{R}^d$

$f \in L^2(D)$

DETERMINISTIC

$$u_\varepsilon - \mathrm{div}\Big(\big(1-\chi_\varepsilon+\varepsilon^2\chi_\varepsilon\big)\nabla u_\varepsilon\Big) = f$$

Energy estimate:

$$\sup_{\omega\in\Omega}\ \sup_{\varepsilon>0}\Big\{ \|u_\varepsilon\|_{L^2(D)} + \|(1-\chi_\varepsilon)\nabla u_\varepsilon\|_{L^2(D;\mathbb{R}^d)} + \|\varepsilon\chi_\varepsilon\nabla u_\varepsilon\|_{L^2(D;\mathbb{R}^d)}\Big\} \le \|f\|_{L^2(D)}$$

$$u_\varepsilon \xrightarrow{\varepsilon'(\omega)} u_0^{(\omega)} \qquad (1-\chi_\varepsilon)\nabla u_\varepsilon \longrightarrow \xi^{(\omega)} \qquad \varepsilon\chi_\varepsilon\nabla u_\varepsilon \longrightarrow 0$$

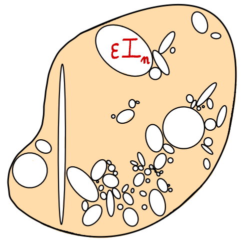
Qualitative homogenization

Thm [B., Duerinckx, Gloria] — HYPOTHESIS ON $\{I_n^{(\omega)}\}_{n\in\mathbb{N}}$

$\mathbb{P}$-a.s.

$$u_\varepsilon^{(\omega)} \xrightarrow[\varepsilon]{L^2(D)} \hat{u} + \mathbb{E}(v)\,(f-\hat{u}) = u_0^{(\omega)} \quad \text{and} \quad (1-\chi_\varepsilon^{(\omega)})\nabla u_\varepsilon^{(\omega)} \xrightarrow[\varepsilon]{L^2(D)} A\nabla\hat{u} = \xi^{(\omega)}$$

where



- $A \in M_{sym}^{d\times d}$: $\quad \lambda\cdot A\lambda = \inf\left\{ \mathbb{E}\left(\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_n I_n}|\lambda+\nabla\varphi|^2\right),\ \begin{array}{l}\varphi\in L^2(\Omega; H^1_{loc}(\mathbb{R}^d)),\\ \nabla\varphi \text{ STATIONARY},\\ \mathbb{E}(\nabla\varphi)=0.\end{array}\right\}$

- $v \in L^2(\Omega; H^1_{loc}(\mathbb{R}^d))$ STATIONARY, $\mathbb{P}$-a.s. SOLUTION OF $\begin{cases} v-\Delta v = 1 & \text{in } \bigcup_n I_n \\ v = 0 & \text{on } \partial I_n,\ \forall n. \end{cases}$

- $\hat{u} \in H^1_0(D)$ SOLUTION OF $(1-\mathbb{E}(v))\hat{u} - \mathrm{div}(A\nabla\hat{u}) = (1-\mathbb{E}(v))f$ in D.

Hypothesis on $\{I_m^{(\omega)}\}_{m\in\mathbb{N}}$

ERGODIC AND STATIONARY

$\left(\exists\ T:\Omega\times\mathbb{R}^d\longrightarrow\Omega \text{ ERGODIC DYNAMICAL SYSTEM}, \atop \exists\ \mathcal{Y}\in\mathcal{A},\ \mathcal{Y}\subset\Omega:\ \bigcup_m I_m(\omega)=\{x\in\mathbb{R}^d:\ T(x)(\omega)\in\mathcal{Y}\}\right)$

$\exists\ 1<p\leq 2:$

$H_1(p)$ $\rightsquigarrow$ EXTENSION PROPERTY

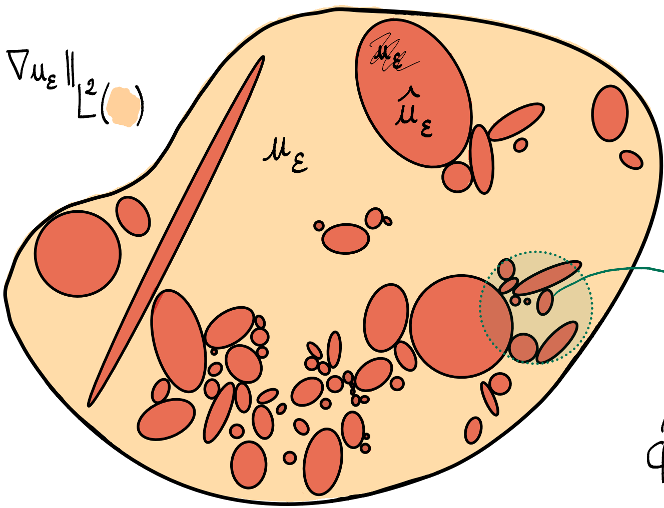
$u_\varepsilon \longrightarrow \hat{u}_\varepsilon$

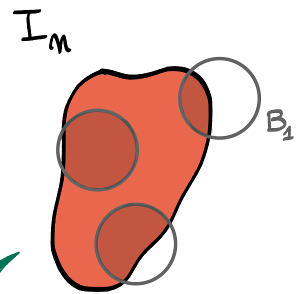
$\mathbb{P}$- a.s.

ELLIPTIC REGULARITY $\leftsquigarrow$ $H_2(p')$

(LOCAL & UNIFORM)

$$\|\nabla\hat{u}_\varepsilon\|_{L^p(\ )}\lesssim_{d,p,D}\|\nabla u_\varepsilon\|_{L^2(\ )}$$





$$\overset{H_0^1(I_m)}{\varphi}-\Delta\varphi=\operatorname{div}(\overset{L^{p'}(B_2)}{G})\quad\text{in } I_m\cap B_2$$

$$\|\nabla\varphi\|_{L^{p'}(I_m\cap B_1)}\lesssim_{d,p'}\|G\|_{L^{p'}(I_m\cap B_2)}+\|\nabla\varphi\|_{L^2(I_m\cap B_2)}$$

Examples of inclusions

$H_1(p)$

- UNIFORMLY LIPSCHITZ INCLUSIONS WITH:

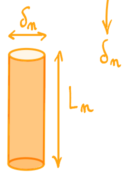
→ UNIFORM SEPARATION $1 < p \leq 2$ ✓

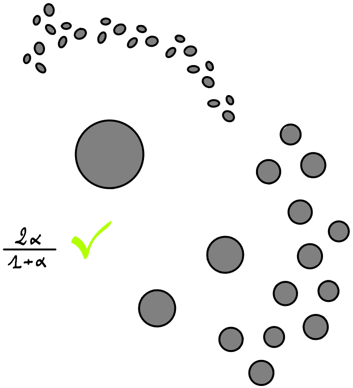
$$\operatorname{diam} I_n \lesssim_d \min_{m \neq n} \operatorname{dist}(I_n ; I_m).$$

→ MOMENT CONDITION ON SEPARATION $1 < p \leq \frac{2\alpha}{1+\alpha}$ ✓

$$\varlimsup_{R \nearrow \infty} \frac{1}{R^d} \sum_{I_n \cap B_R \neq \emptyset} \left(\frac{\operatorname{diam} I_n}{\min_{m \neq n} \operatorname{dist}(I_n ; I_m)} \right)^{\alpha} < +\infty$$

δ_n

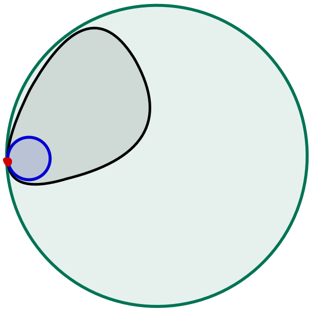




- C^2 STRICTLY CONVEX INCLUSIONS

$$1 \leq \frac{R_n}{r_n} \leq C_*$$

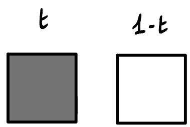
NO SEPARATION CONDITION!



$1 < p \leq \frac{2(d+1)}{d+3}$ ✓

- SUBCRITICAL PERCOLATION CLUSTERS IN RANDOM CHESS STRUCTURE

$1 < p \leq 2$ ✓



$H_2(p')$

- UNIFORMLY C^1 INCLUSIONS (+ modulus of continuity)

$2 \leq p' < +\infty$ ✓

- UNIFORMLY LIPSCHITZ INCLUSIONS

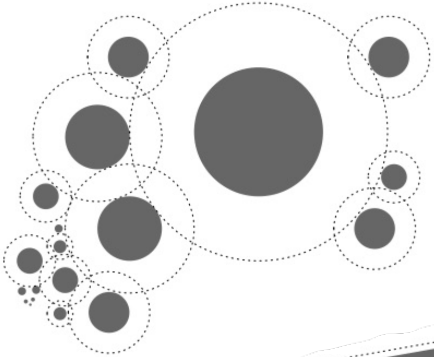
$\exists\, q(d, \text{Lipschitz constant}) > 3$:

$2 \leq p' < q$ ✓

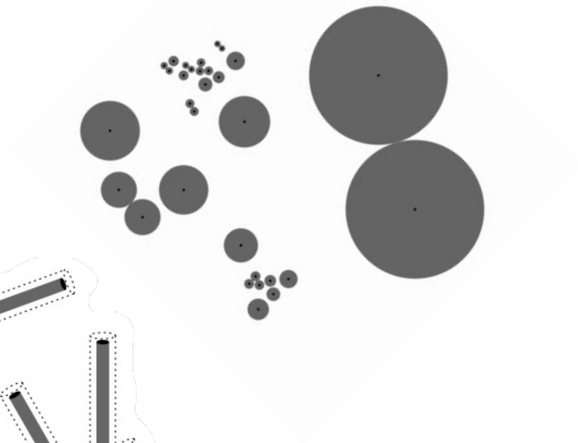
- UNIFORMLY C^1 DEFORMATIONS OF A CONVEX POLYGONAL DOMAIN

$2 \leq p' < +\infty$ ✓

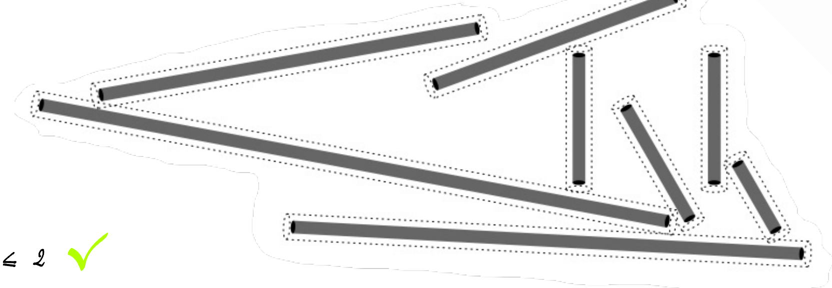
$1 < p \leq \frac{2\alpha}{1+\alpha}$ ✓



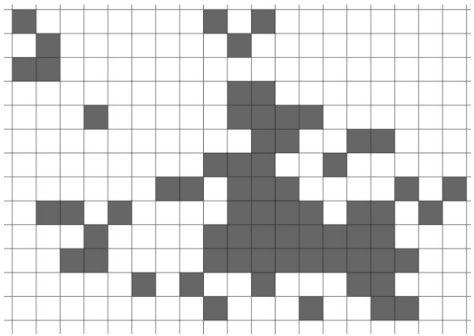
$1 < p \leq \frac{2(d+1)}{d+3}$ ✓



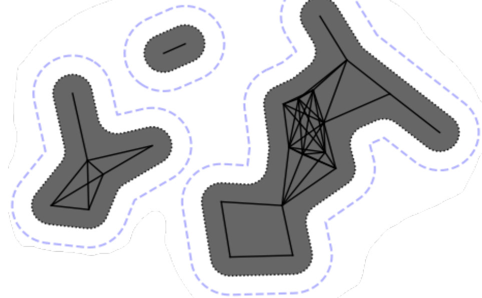
$1 < p \leq \frac{2\alpha}{1+\alpha}$ ✓



$d = 2$ $\quad \frac{3}{2} < p \leq 2$ ✓

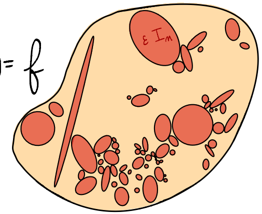


$1 < p \leq 2$ ✓



Arguments of proof in 3 steps

$$u_\varepsilon - \mathrm{div}\left(\left(1-\chi_\varepsilon+\varepsilon^2\chi_\varepsilon\right)\nabla u_\varepsilon\right)=f$$



1) COMPACTNESS FOR FIXED $\omega\in\Omega$, ALONG A SUBSEQUENCE :

$H_1(p)$

$$u_\varepsilon \rightsquigarrow \hat{u}_\varepsilon \in W_0^{1,p}(D), \quad \|\nabla \hat{u}_\varepsilon\|_{L^p(D)} \lesssim \|(1-\chi_\varepsilon)\nabla u_\varepsilon\|_{L^2(D)} \lesssim_f 1 .$$

$$\hat{u}_\varepsilon \xrightarrow{L^p(D)} \hat{u}^{(\omega)}, \qquad \nabla\hat{u}_\varepsilon \xrightarrow{L^p(D)} \nabla\hat{u}$$

CLAIM:

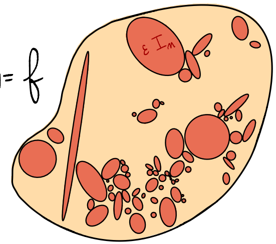
$$\boxed{u_\varepsilon - \hat{u} - v_\varepsilon \overset{L^p(D)}{=} o_\varepsilon(1)}$$

$v_\varepsilon^{(\omega)} \in H_0^1\left(\bigcup_n \varepsilon I_n\right)$ SOLUTION OF $\begin{cases} v_\varepsilon - \varepsilon^2\Delta v_\varepsilon = f-\hat{u} & \text{in } \varepsilon I_n, \\ v_\varepsilon = 0 & \text{on } \varepsilon\partial I_n . \end{cases}$

$$u_\varepsilon - \hat{u} - v_\varepsilon = \underbrace{(1-\chi_\varepsilon)|u_\varepsilon-\hat{u}|}_{\xrightarrow{L^p(D)}\,0} + \underbrace{\chi_\varepsilon(\hat{u}_\varepsilon-\hat{u})}_{\xrightarrow{L^p(D)}\,0} + \underbrace{\chi_\varepsilon\left(u_\varepsilon-\hat{u}_\varepsilon-v_\varepsilon\right)}_{=:\,r_\varepsilon^{(\omega)}} = o_\varepsilon(1) + r_\varepsilon$$

Arguments of proof in 3 steps

$$u_\varepsilon - \operatorname{div}\big((1-\chi_\varepsilon + \varepsilon^2 \chi_\varepsilon)\nabla u_\varepsilon\big) = f$$



1) Compactness for fixed $\omega \in \Omega$, along a subsequence :

$$u_\varepsilon \overset{H_1(p)}{\rightsquigarrow} \hat u_\varepsilon \xrightarrow{W_0^{1,p}(D)} \hat u^{(\omega)}$$

$$\begin{cases} v_\varepsilon - \varepsilon^2 \Delta v_\varepsilon = f - \hat u & \text{in } \varepsilon I_m, \\ v_\varepsilon = 0 & \text{on } \varepsilon \partial I_m. \end{cases}$$

$$u_\varepsilon - \hat u - v_\varepsilon = o_\varepsilon(1) + \underbrace{\chi_\varepsilon\big(u_\varepsilon - \hat u_\varepsilon - v_\varepsilon\big)}_{=:\, r_\varepsilon^{(\omega)} \in W_0^{1,p}(\varepsilon I_m)}$$

Standard elliptic regularity ?

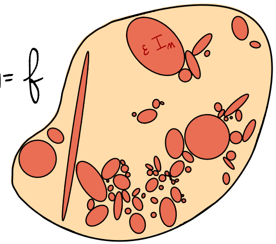
$$r_\varepsilon - \varepsilon^2 \Delta r_\varepsilon = \hat u - \hat u_\varepsilon + \varepsilon \operatorname{div}(\varepsilon \nabla \hat u_\varepsilon) \text{ in } \varepsilon I_m \Longrightarrow \|r_\varepsilon\|_{L^p(\varepsilon I_m)} \lesssim_{d,p} \|\hat u - \hat u_\varepsilon\|_{L^p(\varepsilon I_m)} + \|\varepsilon \nabla \hat u_\varepsilon\|_{L^p(\varepsilon I_m)}$$

$$\begin{cases} r - \Delta r = h & \text{in } I \\ r = 0 & \text{on } \partial I \end{cases} \Longrightarrow \int_I |r|^p + (p-1)|\nabla r|^2 |r|^{p-2} = \int_I h\, |r|^{p-2} r \le \|h\|_p \, \|r\|_p^{p-1}$$

$$|r|^{p-2} r$$

Arguments of proof in 3 steps

$$u_\varepsilon - \operatorname{div}\big((1-\chi_\varepsilon+\varepsilon^2\chi_\varepsilon)\nabla u_\varepsilon\big) = f$$



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$$u_\varepsilon - \hat{u} - v_\varepsilon = o_\varepsilon(1) + \underbrace{\chi_\varepsilon\left(u_\varepsilon - \hat{u}_\varepsilon - v_\varepsilon\right)}_{=:\, r_\varepsilon^{(\omega)}} \in W_0^{1,p}(\varepsilon I_m)$$

Standard elliptic regularity ?

$$r_\varepsilon - \varepsilon^2 \Delta r_\varepsilon = \hat{u} - \hat{u}_\varepsilon + \varepsilon \operatorname{div}(\varepsilon \nabla \hat{u}_\varepsilon) \text{ in } \varepsilon I_m \implies \|r_\varepsilon\|_{L^p(\varepsilon I_m)} \lesssim_{d,p} \overset{o_\varepsilon(1)}{\overset{\|}{\|\hat{u} - \hat{u}_\varepsilon\|_{L^p(\varepsilon I_m)}}} + \|\varepsilon \nabla \hat{u}_\varepsilon\|_{L^p(\varepsilon I_m)}$$

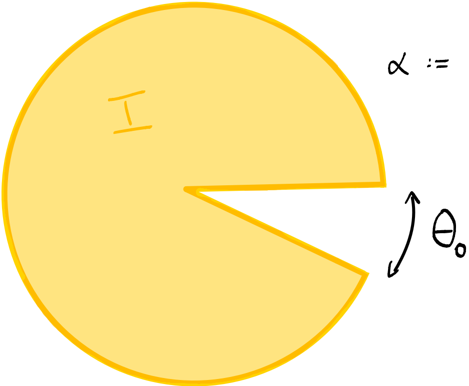
$$\begin{cases} r - \Delta r = \operatorname{div}(H) & \text{in } I \\ r = 0 & \text{on } \partial I \end{cases} \implies \|r\|_p = \sup_{\|h\|_{p'}=1} \left(\int_I r\, h = \int_I \nabla\varphi \cdot H \right)$$

HELMOTZ $\varphi - \Delta\varphi = h$

$$\boxed{\|\nabla\varphi\|_{p'} \overset{?}{\lesssim_I} \|h\|_{p'}}$$

NO !

ϟ REGULARITY OF THE BOUNDED DOMAIN ϟ



$$\alpha := \frac{\pi}{2\pi - \theta_0}$$

$$\varphi(r\cos\theta, r\sin\theta) := r^{\alpha}\sin(\alpha\theta)$$

$$\begin{cases} \Delta\varphi = 0 & \text{in } I \\ \varphi = 0 & \text{on} \end{cases}$$

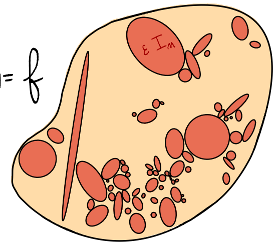
BUT $\nabla\varphi \in L^{p'}(I) \iff p' < \frac{2}{1-\alpha} = 2 + \frac{2\pi}{\pi - \theta_0}$

$$\rightsquigarrow \quad \tilde{\varphi} := \eta\varphi \in C^{\infty}(I), \qquad \begin{cases} \tilde{\varphi} - \Delta\tilde{\varphi} = \underbrace{\boxed{\tilde{\varphi} - 2\nabla\eta\cdot\nabla\varphi - \varphi\Delta\eta}}_{L^{\infty}} & \text{in } I \\ \tilde{\varphi} = 0 & \text{on } \partial I \end{cases}$$

<u>YET</u>, FOR ALL $p' \geq 2 + \frac{2\pi}{\pi - \theta_0}$:

$$\underline{\nabla\tilde{\varphi} = \eta\nabla\varphi + \nabla\eta\,\varphi \notin L^{p'}(I)}$$

Arguments of proof in 3 steps



$$u_\varepsilon - \mathrm{div}\big((1-\chi_\varepsilon+\varepsilon^2\chi_\varepsilon)\nabla u_\varepsilon\big) = f$$

1) Compactness for fixed $\omega \in \Omega$, along a subsequence :

$H_1(p)$

$$u_\varepsilon \rightsquigarrow \hat{u}_\varepsilon \xrightarrow{W_0^{1,p}(D)} \hat{u}^{(\omega)}$$

$$\begin{cases} v_\varepsilon - \varepsilon^2 \Delta v_\varepsilon = f - \hat{u} & \text{in } \varepsilon I_n, \\ v_\varepsilon = 0 & \text{on } \varepsilon \partial I_n. \end{cases}$$

$$u_\varepsilon - \hat{u} - v_\varepsilon = o_\varepsilon(1) + \underbrace{\chi_\varepsilon\left(u_\varepsilon - \hat{u}_\varepsilon - v_\varepsilon\right)}_{=:\, r_\varepsilon^{(\omega)}} \in W_0^{1,p}(\varepsilon I_n)$$

Standard elliptic regularity ?

$$r_\varepsilon - \varepsilon^2 \Delta r_\varepsilon = \hat{u} - \hat{u}_\varepsilon + \varepsilon\, \mathrm{div}(\varepsilon \nabla \hat{u}_\varepsilon) \text{ in } \varepsilon I_n \implies \|r_\varepsilon\|_{L^p(\varepsilon I_n)} \lesssim_{d,p} \underbrace{\|\hat{u} - \hat{u}_\varepsilon\|_{L^p(\varepsilon I_n)}}_{= o_\varepsilon(1)\ \checkmark} + \underbrace{\|\varepsilon \nabla \hat{u}_\varepsilon\|_{L^p(\varepsilon I_n)}}_{= o_\varepsilon(1)\ \checkmark}$$

$H_2(p')$

$$\begin{cases} r - \Delta r = \mathrm{div}(H) & \text{in } I_n \\ r = 0 & \text{on } \partial I_n \end{cases} \implies \|r\|_{L^p(I_n)} \leq C(d,p)\, \|H\|_{L^p(I_n)}$$

Arguments of proof in 3 steps

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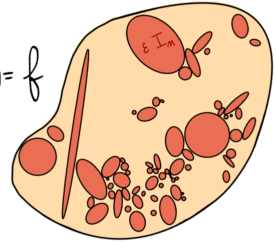


1) Compactness for fixed $\omega \in \Omega$, along a subsequence:

$$u_\varepsilon \rightsquigarrow \hat{u}_\varepsilon \xrightarrow[H_1(p)]{W_0^{1,p}(D)} \hat{u}^{(\omega)}$$

$$\begin{cases} v_\varepsilon - \varepsilon^2 \Delta v_\varepsilon = f - \hat{u} & \text{in } \varepsilon I_m, \\ v_\varepsilon = 0 & \text{on } \varepsilon \partial I_m. \end{cases}$$

$H_2(p')$

$$u_\varepsilon - \hat{u} - v_\varepsilon = o_\varepsilon(1) \quad \text{in } L^p(D)$$

2) From ergodic thm inside inclusions to resonant asymptotics:

$\exists!\ v \in L^2(\Omega; H^1_{loc}(\mathbb{R}^d))$ stationary, $\mathbb{P}$-a.s. solution of $\begin{cases} v^{(\omega)} - \Delta v^{(\omega)} = 1 & \text{in } I_m^{(\omega)} \\ v^{(\omega)} = 0 & \text{on } \partial I_m^{(\omega)} \end{cases}$

Claim:

$$\boxed{v_\varepsilon^{(\omega)} = v^{(\omega)}\left(\tfrac{\cdot}{\varepsilon}\right)\left(f - \hat{u}\right) + o_\varepsilon(1) \quad \text{in } L^2(D)}$$

$+$

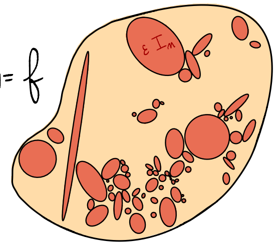
Ergodic thm

$$v^{(\omega)}\left(\tfrac{\cdot}{\varepsilon}\right) \xrightarrow{L^2(D)} \mathbb{E}(v)$$

$$\Longrightarrow \quad \begin{array}{ccc} u_\varepsilon^{(\omega)} & \xrightarrow{\mathcal{D}'(D)} & \hat{u}^{(\omega)} + \mathbb{E}(v)\left(f - \hat{u}\right) \\ \Big\downarrow {\scriptstyle L^2(D)} & & \\ u_0^{(\omega)} & & \end{array} \quad \Longrightarrow \quad u_0^{(\omega)} = \hat{u}^{(\omega)} + \mathbb{E}(v)\left(f - \hat{u}\right)$$

Arguments of proof in 3 steps

$$u_\varepsilon - \mathrm{div}\big((1 - \chi_\varepsilon + \varepsilon^2 \chi_\varepsilon)\nabla u_\varepsilon\big) = f$$



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$$u_\varepsilon \overset{H_1(p)}{\rightsquigarrow} \hat{u}_\varepsilon \xrightarrow{W_0^{1,p}(D)} \hat{u}^{(\omega)}$$

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$H_2(p')$

$$u_\varepsilon - \hat{u} - v_\varepsilon = o_\varepsilon(1) \quad \text{in } L^p(D)$$

2) FROM ERGODIC THM INSIDE INCLUSIONS TO RESONANT ASYMPTOTICS :

$$\begin{cases} v_\varepsilon \overset{L^2(D)}{\sim} v\left(\frac{\cdot}{\varepsilon}\right)(f - \hat{u}) \\ u_0^{(\omega)} = \hat{u}^{(\omega)} + \mathbb{E}(v)(f - \hat{u}) \end{cases}$$

3) IDENTIFICATION OF $\hat{u}^{(\omega)}$ AND INDEPENDANCE WRT SUBSEQUENCE VIA TARTAR'S OSCILLATION TEST FUNCTIONS :

$H_1(p)$

CORRECTORS:

$\exists\ \varphi_i \in L^p(\Omega; W^{1,p}_{loc}(\mathbb{R}^d))$

$$\begin{cases} \mathbb{P}\text{-a.s. SOLUTION OF } -\mathrm{div}\big(\mathbb{1}_{\mathbb{R}^d \setminus \bigcup_n I_n}(e_i + \nabla \varphi_i)\big) = 0 \text{ in } \mathbb{R}^d, \\ \nabla \varphi_i \text{ STATIONARY, } \mathbb{E}(\nabla \varphi_i) = 0 . \end{cases}$$

NB: $\mathbb{1}_{\mathbb{R}^d \setminus \bigcup_n I_n} \nabla \varphi_i \in L^2(\Omega; H^1_{loc}(\mathbb{R}^d \setminus \bigcup_n I_n))$ UNIQUELY DEFINED, $\quad \lambda \cdot A \lambda = \mathbb{E}\left(\left|\mathbb{1}_{\mathbb{R}^d \setminus \bigcup_n I_n}(\lambda + \nabla \varphi_\lambda)\right|^2\right)$

$$\text{TARTAR}: \quad \psi \in C_c^\infty(D) \rightsquigarrow \eta_\varepsilon^{(\omega)} := \psi + \varepsilon \sum_{i=1}^{d} \varphi_i^{(\omega)}\left(\tfrac{\cdot}{\varepsilon}\right) \partial_i \psi$$

DOUBLE-POROSITY PDE:

$$\int_D u_\varepsilon \eta_\varepsilon + (\varepsilon^2 \chi_\varepsilon + 1 - \chi_\varepsilon) \nabla u_\varepsilon \cdot \nabla \eta_\varepsilon = \int_D f \eta_\varepsilon \longrightarrow \int_D f \psi$$

$$\|$$

$$\int_D u_\varepsilon \psi + \int_D (\cancel{\varepsilon^2 \chi_\varepsilon} + 1 - \chi_\varepsilon) \nabla u_\varepsilon \cdot \left(\sum_{i=1}^{d} \left(e_i + \nabla \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)\right) \partial_i \psi + \cancel{\varepsilon \sum_i \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) \nabla \partial_i \psi} \right) + o_\varepsilon(1)$$

$$\downarrow$$

$$u_0$$

$\|$ IPP

$$\int_D u_0 \psi + \int_D \sum_{i=1}^{d} \left(\cancel{\nabla (u_\varepsilon \partial_i \psi)} - \underset{\sim\, \hat{u}}{u_\varepsilon} \nabla \partial_i \psi \right) \cdot \underbrace{(1 - \chi_\varepsilon)\left(e_i + \nabla \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)\right)}_{\xrightarrow{L^2(D)} \mathbb{E}\left(\mathbb{1}_{\mathbb{R}^d \setminus \cup_m I_m} (e_i + \nabla \varphi_i)\right) = A e_i} + o_\varepsilon(1)$$

$\|$ IPP

$$\int_D u_0 \psi + \int_D \nabla \hat{u} \cdot A \nabla \psi + o_\varepsilon(1) \longrightarrow \int_D (1 - \mathbb{E}(v)) \hat{u} \psi + A \nabla \hat{u} \cdot \nabla \psi - \int_D \mathbb{E}(v) f \psi$$

$$\Longrightarrow \boxed{\hat{u}^{\cancel{(\omega)}} = \hat{u} \in H_0^1(D) \text{ THE SOLUTION OF } (1 - \mathbb{E}(v)) \hat{u} - \operatorname{div}(A \nabla \hat{u}) = (1 - \mathbb{E}(v)) f \text{ in } D.}$$

$$\text{TARTAR}: \quad \psi \in C_c^\infty(D) \rightsquigarrow \eta_\varepsilon^{(\omega)} := \psi + \varepsilon \sum_{i=1}^{d} \varphi_i^{(\omega)}\left(\tfrac{\cdot}{\varepsilon}\right) \partial_i \psi \quad \cancel{\in H_0^1(D)}$$

$$\varphi_i, \nabla\varphi_i \in L^{\cancel{2}\,p}(I_m)$$

DOUBLE-POROSITY PDE:

$$\int_D u_\varepsilon \eta_\varepsilon + (\varepsilon^2 \chi_\varepsilon + 1 - \chi_\varepsilon)\nabla u_\varepsilon \cdot \nabla \eta_\varepsilon = \int_D f \eta_\varepsilon \xrightarrow{?} \int_D f \psi$$

$$\|$$

$$\underset{\downarrow \; u_0}{\int_D u_\varepsilon \psi} + \int_D (\varepsilon^2 \chi_\varepsilon + 1 - \chi_\varepsilon)\nabla u_\varepsilon \cdot \left(\sum_{i=1}^{d} \left(e_i + \nabla \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)\right)\partial_i \psi + \varepsilon \sum_i \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) \nabla \partial_i \psi \right) + o_\varepsilon(1) \quad ?$$

$\varphi_{\varepsilon,i} \in L^2(\Omega; H^1_{loc}(\mathbb{R}^d))$ STATIONARY, $\mathbb{P}$-a.s. SOLUTION OF

$$\varepsilon^2 \varphi_{\varepsilon,i} - \operatorname{div}\left((\varepsilon^2 \chi_\varepsilon + 1 - \chi_\varepsilon)(e_i + \nabla\varphi_{\varepsilon,i})\right) = 0 \quad \text{in } \mathbb{R}^d.$$

Thank you for your attention!