# Good Lie Brackets for quantum harmonic oscillator

**Andrei AGRACHEV**, SISSA - Trieste **Bettina KAZANDJIAN**, LJLL - Paris
**Eugenio POZZOLI**, IRMAR - Rennes

We study the Schrödinger's equation of a quantum harmonic oscillator

$$\frac{d}{dt}\psi_t(x) = \frac{i}{2}\Delta\psi_t(x) + u(t)\frac{i|x|^2}{2}\psi_t(x), \qquad \psi_t \in L^2(\mathbb{R}^n, \mathbb{C}), u \in \mathrm{PWC}([0,T], \mathbb{R}). \tag{1}$$

*Definition* 1. **STAR operator.** Let $L$ be a linear bounded operator on $L^2(\mathbb{R}^n, \mathbb{C})$. For every initial condition $\varphi_0 \in L^2$ we denote $\psi(t; u, \varphi_0)$ the solution of (1) at time $t$ with $u \in \mathrm{PWC}([0,t], \mathbb{R})$ and initial condition $\varphi_0$. The operator $L$ is small-time approximately reachable (STAR) by the system (1) if for every $\varphi_0 \in L^2$, for every $\varepsilon > 0$ and $T > 0$, there exist $\tau \in [0, T]$ and $u : [0, \tau] \to \mathbb{R} \in \mathrm{PWC}([0,\tau], \mathbb{R})$ such that

$$||L\varphi_0 - \psi(\tau; u, \varphi_0)||_{L^2} < \varepsilon.$$

In the work [1], we describe the small-time reachable orbits for the Schrödinger's system thanks to the unitary representation of $SL_2(\mathbb{R})$ or one of its covering spaces. In fact, it can be observed that the algebra of linear operators Lie $\{\frac{i|x|^2}{2}, \frac{i\Delta}{2}\}$ equipped with the commutator is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Then the study of the small-time controllability problem on the finite-dimensional Lie groups $SL_2(\mathbb{R})$ and its covering spaces can be applied to the infinite-dimensional problem, and leads to the following result.

*Theorem* 1. **STAR orbits of (1).** For every $\alpha, \sigma \in \mathbb{R}$ and $\beta > 0$, the operators $e^{i\alpha|x|^2}$, $e^{i\sigma\Delta}$ and $D_\beta$ are small-time approximately reachable by system (1), where

$$(D_\beta\varphi)(x) := \beta^{\frac{n}{2}}\varphi(\beta x).$$

Such orbits have recently been found by Beauchard and the third author using an analytical approach, and have been used to construct the first examples of small-time globally approximately controllable bilinear Schrödinger PDEs, see e.g. [2]. Similar analysis were performed since the early days of quantum control, see e.g. [3]. Here we present an alternative approach where recent advances of finite-dimensional geometric control theory can be applied directly to control the Schrödinger PDE of a quantum harmonic oscillator, thanks to the specific nature of this infinite-dimensional system.

[1] A. Agrachev, B. Kazandjian, E. Pozzoli. *Good lie brackets for classical and quantum harmonic oscillators.* preprint, hal-04991273, 2025.

[2] K. Beauchard, E. Pozzoli. *Examples of small-time controllable schrödinger equations.* Annales Henri Poincaré, 2025.

[3] K. Mirrahimi, P. Rouchon. *Controllability of quantum harmonic oscillators.* IEEE Trans. Automat. Control, 2004.

Contact : bettina.kazandjian@sorbonne-universite.fr