One dimensional approximation of measures in Wasserstein distance

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In this talk we will discuss a variational problem recently introduced in [2], which consists of approximating a given probability measure ρ_0 with a measure uniformly distributed over connected sets with finite length. This is done by minimizing the Wasserstein distance between ρ_0 and all such 1-dimensional measures with a penalization of the total length of the support, being therefore closely related to the so-called *average distance minimization problem* [1]. Due to concentration of mass effects, it seems intractable to show existence of an optimal set by means of the Direct Method. For this reason we introduce a relaxation of the notion of length of the support of a probability measure, being therefore able to show existence to this relaxed formulation. Fortunately, it has the property that if ρ_0 does not give mass to one dimensional sets, any minimizer of the relaxation is a minimizer of the original problem.
In the sequel we shall discuss some qualitative properties of this problem as Ahlfors regularity, absence of loops and if time permits a phase field approximation[3].

[1] G. Buttazzo, E. Stepanov. *Optimal transportation networks as free dirichlet regions for the monge-kantorovich problem.* Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, **2(4)**, 631–678, 2003.

[2] A. Chambolle, V. Duval, J. M. Machado. *One-dimensional approximation of measures in Wasserstein distances.* Journal de l'École Polytechnique, 2025.

[3] J. M. Machado. *Phase-field approximation for 1-dimensional shape optimization problems.* accepted in SIAM journal of Mathematical Analysis, 2025.

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