Quadrature optimale sur des Cubed Spheres de faible résolution

Optimal quadrature rule on low-resolution Cubed Spheres

Jean-Baptiste Bellet† Matthieu Brachet†
Jean-Pierre Croisille‡

†Laboratoire de Mathématiques et Applications, Université de Poitiers, CNRS, F-86073 Poitiers, France
‡Université de Lorraine, CNRS, IECL, F-57000 Metz, France

Congrès SMAI 2025
3 juin 2025, Carcans Maubuisson

Framework

Approximation on Spherical Grids using Spherical Harmonics

- Data approximation on the sphere, such as interpolation
- Spectral solvers for PDEs on the sphere, *e.g.* the advection equation
  - M. Brachet's talk

Supported by Les Enveloppes Fluides et l'Environnement

- LEFE-MANU call, Section "Océan-Atmosphère",
- from the French National program AAP-INSU2024

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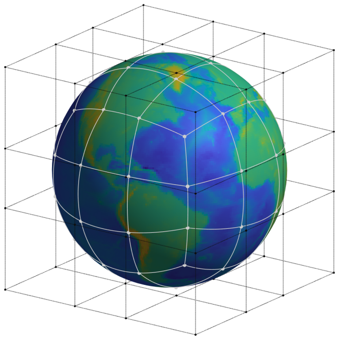
Equiangular Cubed Sphere

$$\mathrm{CS}_N := \Big\{\rho(\pm 1, u, v),\ \rho(u, \pm 1, v),\ \rho(u, v, \pm 1);$$
$$u = \tan\tfrac{i\pi}{2N},\ v = \tan\tfrac{j\pi}{2N},\ -\tfrac{N}{2} \le i, j \le \tfrac{N}{2}\Big\}, \quad \rho(x) = \tfrac{x}{|x|}$$



- Angular step: $\frac{\pi}{2N}$
- $|\mathrm{CS}_N| = 6N^2 + 2$ nodes

Fig. Equiangular arcs of great circles on $\mathbb{S}^2$, obtained by radial projection of a meshed circumscribed cube.

R. Sadourny, Conservative finite-difference approximations of the primitive equations on quasi-uniform spherical grids, Monthly Weather Review, 100 (1972), pp. 136-144.

J.-B. Bellet, Symmetry group of the equiangular cubed sphere, Quarterly of Applied Mathematics, 80 (2022), pp. 69-86.

Some quadrature rules on the Cubed Sphere

$$\int_{\mathbb{S}^2} f(x)\,\mathrm{d}\sigma \approx \sum_{x\in \mathrm{CS}_N} \omega(x) f(x) =: Q_\omega(f), \quad \omega : \mathrm{CS}_N \to \mathbb{R}$$

- Bivariate angular trapezoidal rules on the 6 faces
  - 4th/6th order of numerical accuracy if corrected on the 8 corners

M. Brachet, Schémas compacts hermitiens sur la Sphère: applications en climatologie et océanographie numérique, PhD thesis, Université de Lorraine, 2018.

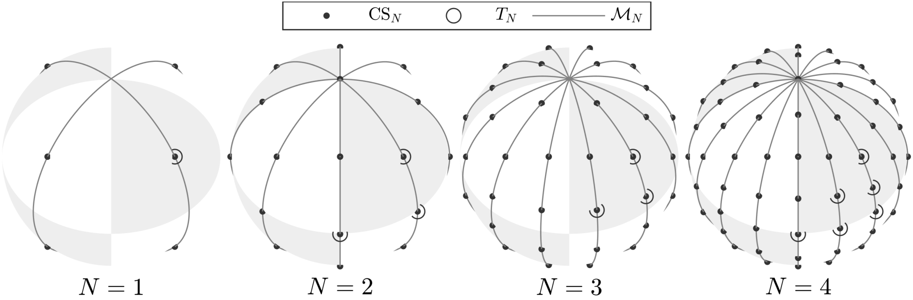
- Interpolatory quadrature rules (with numerical linear algebra)
  - Numerical degree of accuracy

$$2N+1, \text{ if } N = 2p+1 \neq 3,$$
$$2N+3, \text{ if } N = 2p \neq 4,$$
$$4N-1, \text{ if } 1 \leq N \leq 4.$$

  - Extra-accuracy with $N = 3, 4$

J.-B. Bellet, M. Brachet, and J.-P. Croisille, Quadrature and symmetry on the Cubed Sphere, Journal of Computational and Applied Mathematics, 409 (2022).

Focus on the Cubed Sphere CS_N with $1 \leq N \leq 4$



Lemma (lemma of the meridians)

For $1 \leq N \leq 4$, the grid CS_N is included in a set of equiangular meridians,

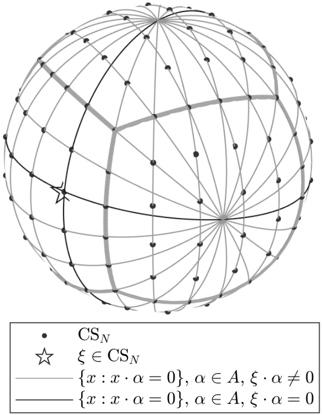
$$\mathrm{CS}_N \subset \mathcal{M}_N := \{x(\theta, \phi), \ \textit{with} \ -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}, \quad \phi \equiv \tfrac{\pi}{4} \ [\tfrac{\pi}{2N}]\},$$

with $x(\theta, \phi) = (\cos\theta \cos\phi, \cos\theta \sin\phi, \sin\theta), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \ \phi \in \mathbb{R}.$

Great circles and Lagrange interpolation on CS_N

Theorem (Lagrange interpolation on CS_N with degree $4N-1$)

Let $N \geq 1$. For every $y : \mathrm{CS}_N \to \mathbb{R}$, there exists $f \in \mathcal{P}_{4N-1}$ which interpolates y, i.e. $f(x) = y(x)$, $x \in \mathrm{CS}_N$.



Proof. Lagrange-kind basis:

$$L_\xi(x) = \frac{1 + \xi \cdot x}{2} \prod_{\substack{\alpha \in A \\ \xi \cdot \alpha \neq 0}} \frac{x \cdot \alpha}{\xi \cdot \alpha} \in \mathcal{P}_{4N-1};$$

$$L_\xi(\xi') = \delta_\xi(\xi'), \; \xi, \xi' \in \mathrm{CS}_N.$$

□

J.-B. Bellet. Mathematical and numerical methods for three-dimensional reflective tomography and for approximation on the sphere. Habilitation thesis, Université de Lorraine, 2023.

Main result: optimal rule on low-resolution Cubed Spheres

Recall that $Q_\omega(f) = \sum_{x \in \mathsf{CS}_N} \omega(x) f(x), \quad \omega : \mathsf{CS}_N \to \mathbb{R}$.

Theorem (Optimal quadrature rule on CS_N, $1 \leq N \leq 4$)

Fix $1 \leq N \leq 4$, *and* $\omega_0(x) = \int_{\mathbb{S}^2} L_x(y)\, \mathrm{d}\sigma$, $x \in \mathsf{CS}_N$.

1. *The weight ω_0 is positive, has octahedral symmetry, and is given in the next slide.*
2. *The quadrature rule Q_{ω_0} has the degree of accuracy $4N - 1$.*
3. *The rule Q_{ω_0} is the optimal one, i.e. any rule Q_ω with $\omega \neq \omega_0$ has a smaller degree of accuracy.*

J.-B. Bellet, M. Brachet, and J.-P. Croisille, Quadrature on the Cubed Sphere: the low-resolution case, hal-04807672 (2024).

Analytical expression of the optimal quadrature weights

N	x_1	x_2	x_3	$\omega_0(x_1, x_2, x_3)$	$\lvert CS_N \rvert$	Degree
1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{\pi}{2}$	8	3
2	$\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{2}}$ 1	$\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{2}}$ 0	$\frac{1}{\sqrt{3}}$ 0 0	$\frac{9\pi}{70}$ $\frac{16\pi}{105}$ $\frac{4\pi}{21}$	26	7
3	$\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{2+t^2}}$ $\frac{1}{\sqrt{1+2t^2}}$	$\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{2+t^2}}$ $\frac{t}{\sqrt{1+2t^2}}$	$\frac{1}{\sqrt{3}}$ $\frac{t}{\sqrt{2+t^2}}$ $\frac{t}{\sqrt{1+2t^2}}$	$\frac{9\pi}{140}$ $\frac{61\pi}{840} - \frac{3\pi\sqrt{3}}{560}$ $\frac{61\pi}{840} + \frac{3\pi\sqrt{3}}{560}$ with $t = 2 - \sqrt{3}$	56	11
4	$\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{2+s^2}}$ $\frac{1}{\sqrt{1+2s^2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{1+s^2}}$ 1	$\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{2+s^2}}$ $\frac{s}{\sqrt{1+2s^2}}$ $\frac{1}{\sqrt{2}}$ $\frac{s}{\sqrt{1+s^2}}$ 0	$\frac{1}{\sqrt{3}}$ $\frac{s}{\sqrt{2+s^2}}$ $\frac{s}{\sqrt{1+2s^2}}$ 0 0 0	$\frac{729\pi}{20020}$ $\frac{2053\pi}{51480} - \frac{183\pi\sqrt{2}}{80080}$ $\frac{2053\pi}{51480} + \frac{183\pi\sqrt{2}}{80080}$ $\frac{512\pi}{15015}$ $\frac{2048\pi}{45045}$ $\frac{736\pi}{15015}$ with $s = \sqrt{2} - 1$	98	15

Comments on the rules

- Same rules, with a new approach, without rounding errors, as

  J.-B. Bellet, M. Brachet, and J.-P. Croisille, Quadrature and symmetry on the Cubed Sphere, Journal of Computational and Applied Mathematics, 409 (2022)

- $N = 1$: 8 nodes, degree 3 (cube)
  - uniform rule, corresponding to areas of the projected octahedron
  - tensor product, trapezoidal rule in ϕ times Gauss-Legendre rule in x_3

    K. Atkinson and W. Han. *Spherical harmonics and approximations on the unit sphere: an introduction*, volume 2044. Springer Science & Business Media, 2012.

- $N = 2$: 26 nodes, degree 7 (cube, octahedron, cuboctahedron)
  - available in double precision as a Lebedev's rule in

    J. Burkardt. Sphere Lebedev Rule. *Online, last revised on 2010.*
    https://people.math.sc.edu/Burkardt/c_src/sphere_lebedev_rule/sphere_lebedev_rule.html

- $N = 3$: 56 nodes, degree 11, $N = 4$: 98 nodes, degree 15
  - Exact formulas that seem new

Proof, part I

Assume that $\omega : \mathrm{CS}_N \to \mathbb{R}$ is such that Q_ω is exact on $\mathcal{P}_{4N-1}$. Compute ω.

a. $\omega = \omega_0$: $\omega(x) = Q_\omega(L_x) = \int_{\mathbb{S}^2} L_x \,\mathrm{d}\sigma = \omega_0(x), \; x \in \mathrm{CS}_N$
b. octahedral invariance: for any octahedral symmetry R, $\omega(Rx) = Q_\omega(L_x(R^\intercal\cdot)) = \int_{\mathbb{S}^2} L_x(R^\intercal\cdot)\,\mathrm{d}\sigma = \int_{\mathbb{S}^2} L_x \,\mathrm{d}\sigma = \omega(x)$, $x \in \mathrm{CS}_N$
c. computation of ω: for any $z = x(\theta_z, \phi_z) \in T_N$,

$$\omega(z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} \underbrace{L_z(x(\theta,\phi))}_{\text{trigo. polynom., deg. } \leq 4N-1} \mathrm{d}\phi}_{\text{trapezoidal rule in } \phi,\text{ step } \frac{\pi}{2N}} \cos\theta \mathrm{d}\theta$$

$$= \frac{\pi}{2N} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{[L_z(x(\theta,\phi_z) + L_z(x(\theta,\phi_z+\pi))]\cos\theta}_{\text{trigo. polynom., deg. } \leq 4N} \,\mathrm{d}\theta$$

$$= \cdots \quad \text{(symbolic computation in Maple)}$$

d. ω is positive: observation

Proof, part II

Consider the weight ω_0 in the table, extended to CS_N by octahedral invariance. We check that Q_{ω_0} is exact on $\mathcal{P}_{4N-1}$.

- (a) Q_{ω_0} is exact on $\mathcal{P}_{4N-1}$, if, and only if, it is exact on the following invariant polynomials,

N	Polynomials $v_1^\alpha v_2^\beta$ with degree $4\alpha + 6\beta \leq 4N - 1$
1	1
2	1, v_1, v_2
3	1, v_1, v_2, v_1^2, $v_1 v_2$
4	1, v_1, v_2, v_1^2, $v_1 v_2$, v_1^3, v_2^2, $v_1^2 v_2$

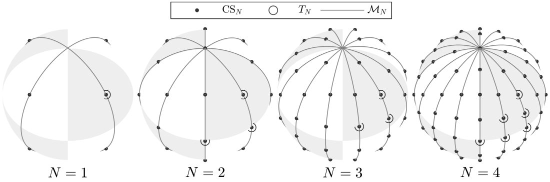
$$v_1 = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2, \quad v_2 = x_1^2 x_2^2 x_3^2$$

V. I. Lebedev. Quadratures on a sphere. *USSR Computational Mathematics and Mathematical Physics*, 16(2):10–24, 1976.

- (b) Check by symbolic computation in Maple

Proof, part III

The degree of accuracy $4N - 1$ is optimal due to a counterexample of degree $4N$.



- (a) Consider a polynomial $p \in \mathcal{P}_{2N}$ such that $p|_{\mathcal{M}_N} = 0$,

$$p(x(\theta, \phi)) = Y_{2N}^{-2N}(x(\theta, \phi - \tfrac{\pi}{4})) = C \cos^{2N} \theta \sin 2N(\phi - \tfrac{\pi}{4}),$$

  with C a constant such that $\|p\|_{L^2}^2 = 1$.
- (b) Due to the lemma of the meridians, $\mathrm{CS}_N \subset \mathcal{M}_N$, so $p|_{\mathrm{CS}_N} = 0$.
- (c) Then, $p^2 \in \mathcal{P}_{4N}$, $\int_{\mathbb{S}^2} p^2 \,\mathrm{d}\sigma = 1$, but $\forall \omega : \mathrm{CS}_N \to \mathbb{R}$, $Q_\omega(p^2) = 0$.

J.-B. Bellet and J.-P. Croisille, Least Squares Spherical Harmonics Approximation on the Cubed Sphere, Journal of Computational and Applied Mathematics, 429 (2023).

About the linear system satisfied by the weight

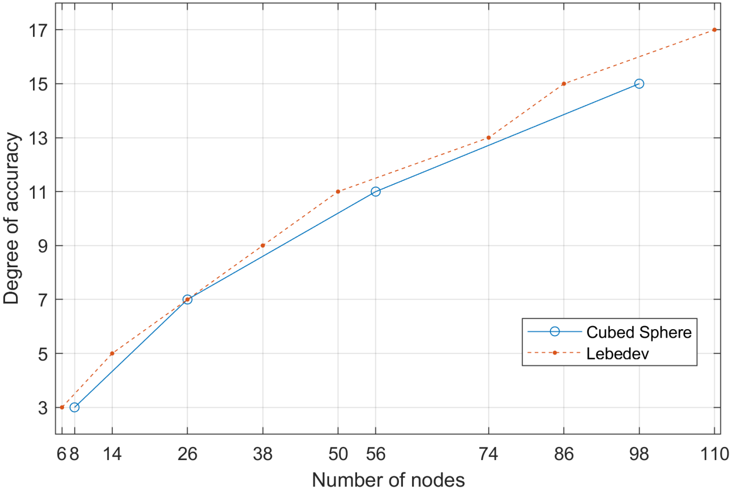
- The octahedral rule Q_{ω_0} integrates the listed invariant polynomials.
- This gives a linear system satisfied by the weights $\omega_0(x)$, $x \in T_N$.

Reduced linear system satisfied by the octahedral weight

N	Number of equations	Number of unknowns
1	1	1
2	3	3
3	5	3
4	8	6

- Extra-accuracy for $N = 3, 4$

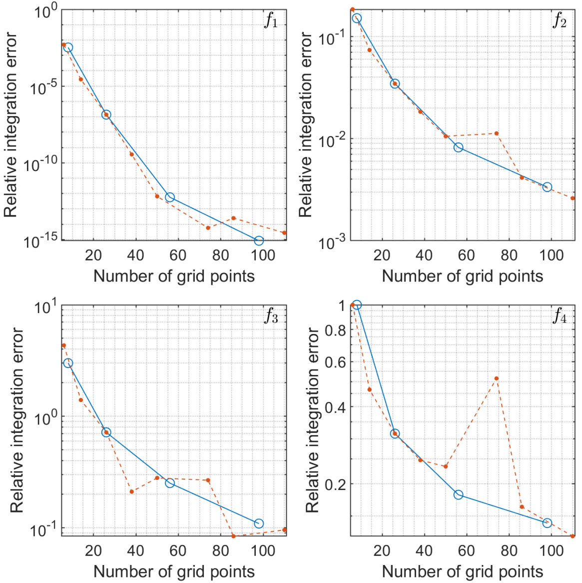
Comparison with Lebedev's octahedral rules

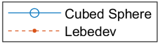


V. I. Lebedev. Values of the nodes and weights of ninth to seventeenth order Gauss-Markov quadrature formulae invariant under the octahedron group with inversion. *USSR Computational Mathematics and Mathematical Physics*, 15(1):44–51, 1975.

J. Burkardt. Sphere Lebedev Rule. *Online (...)*

Integration of test functions





i	$f_i(x, y, z)$
1	$\exp(x)$
2	$\frac{3}{4}\exp\left[-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4} - \frac{(9z-2)^2}{4}\right] + \frac{3}{4}\exp\left[-\frac{(9x+1)^2}{49} - \frac{9y+1}{10} - \frac{9z+1}{10}\right]$ $\frac{1}{2}\exp\left[-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4} - \frac{(9z-5)^2}{4}\right] - \frac{1}{5}\exp\left[-\frac{(9x-4)^2}{1} - \frac{(9y-7)^2}{1} - \frac{(9z-5)^2}{1}\right]$
3	$\cos(3\arccos z) \times \mathbf{1}(3\arccos z \leq \frac{\pi}{2})$
4	$\mathbf{1}(z \geq \frac{1}{2})$

(Maximum error on 1000 random orthogonal transformations of the grids)

Conclusion

Optimal quadrature rule on CS_N, $1 \leq N \leq 4$

- $6N^2 + 2$ nodes, degree of accuracy $4N - 1$
- Simple approach, based on the specific geometry of the grid
- Not so far from Lebedev's octahedral rules
- Application: weighted least squares

$$\inf_{f \in \mathcal{Y}_{2N-1}} \sum_{x \in \mathsf{CS}_N} \omega(x) |f(x) - y(x)|^2, \quad y : \mathsf{CS}_N \to \mathbb{R}$$

Open questions

- Further explanation of the extra-accuracy for $N = 3, 4$?
- Tiling of the sphere with areas given by the quadrature weights ?
- Optimal quadrature rule on CS_N with $N \geq 5$?