Computer assisted analysis of steady states and stability in nonlinear diffusion models

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Introduction

Ambition: be capable to "fully" analyse a dynamic model.

Genesis of the work: Existence of many steady states in nonlinear cross-diffusion model [BP24]

Joint work with Maxime Breden, Olivier Hénot and Antoine Zurek

A general dynamics

PDE

(1)
$$\begin{cases} \partial_t u = F(u), & u = u(t,x) \in \mathbb{R},\ x \in \Omega = (0,1), \\ \partial_n u = 0, & x \in \partial\Omega, \end{cases}$$

- $F(u) = \partial_x(\mathcal{A}(u)\nabla u) + \mathcal{R}(u)$, $H^2(\Omega,\mathbb{R}) \longrightarrow L^2(\Omega,\mathbb{R})$
- $\mathcal{A}(u)$ is the diffusion matrix, $H^2(\Omega,\mathbb{R}) \longrightarrow H^1(\Omega,\mathbb{R})$
- $\mathcal{R}(u)$ is the reaction term, $L^2(\Omega,\mathbb{R}) \longrightarrow L^2(\Omega,\mathbb{R})$.

Let $H^2_N(\Omega,\mathbb{R}) = \{u \in H^2(\Omega,\mathbb{R}),\ \partial u = 0 \text{ on } \partial\Omega\}$.

Overview on Stability

The linearized equation around the steady state $\tilde{u}$ is:

$$(2) \qquad \begin{cases} \partial_t v = \mathcal{L}(\tilde{u})v, \\ v|_{t=0} = v^0 \in H^2_N(\Omega, \mathbb{R}), \end{cases} \qquad \text{where } \mathcal{L}(\tilde{u}) = DF(\tilde{u}).$$

Lyapunov Functional Strategy

Strategy: aiming to find a positive-definite self-adjoint operator P such that there exists $\mu > 0$ and:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Pv, v\rangle_{L^2(\Omega,\mathbb{R})} = \langle (P\mathcal{L}(\tilde{u}) + \mathcal{L}(\tilde{u})^* P)v, v\rangle_{L^2(\Omega,\mathbb{R})} \leq -\mu \|v\|^2_{L^2(\Omega,\mathbb{R})},$$

for any solution v to the linearized equation.

Theorem on Stability

Theorem

Let P be positive and self-adjoint. Define $Q = -(P\mathcal{L}(\tilde{u}) + \mathcal{L}(\tilde{u})^* P)$. If $Q \geq \mu$, with $\mu > 0$, then there exists $\lambda_0 > 0$ such that:

$$\sigma(\mathcal{L}(\tilde{u})) \subset \{z \in \mathbb{C}, \mathrm{Re}(z) < -\lambda_0\}.$$

Furthermore, there exists $\varepsilon > 0$ such that $\|u^0 - \tilde{u}\|_{L^2(\Omega,\mathbb{R})}$ implies

$$\|u(t,\cdot) - \tilde{u}\|_{L^2(\Omega,\mathbb{R})} \underset{t\to+\infty}{\longrightarrow} 0, \ \textit{exponentially}$$

with $u|_{t=0} = u^0 \in H^2_N(\Omega,\mathbb{R})$.

Proof.

The main ideas comes from [Dat70]. □

Framework

Fourier Series Representation

For $\Omega = (0,1)$, we seek an isolated stationary solution u as the Fourier series:

$$u(x) = \sum_{k \in \mathbb{N}} u_k \cos(\pi k \cdot x),$$

which belongs to $H^2_N(\Omega, \mathbb{R})$.

Sequence Spaces

Let $p = 1, 2$, we define

$$\ell^p = \left\{ u \in \mathbb{R}^{\mathbb{N}}, \|u\|_{\ell^p} < +\infty \right\},$$

where

$$\|u\|_{\ell^1} = \sum_{k \in \mathbb{N}} |u_k|,$$

$$\|u\|^2_{\ell^2} = u_0^2 + \frac{1}{2} \sum_{k \in \mathbb{N}^*} u_k^2.$$

Projection on finite spaces: $N \in \mathbb{N}$, $\pi^{\leq N} u := (u_k)_{k=0,\ldots,N}$.

Remark: $(\ell^1, +, \star, \|\cdot\|_{\ell^1})$ is a Banach algebra, where $\star$ is the discrete convolution on $\mathbb{R}^{\mathbb{N}}$. $(\ell^2, +, \langle \cdot, \cdot \rangle_{\ell^2})$ is an Hilbert space.

Consider $G : u \to u - AF(u)$, with A is an injective linear operator such that $A\mathcal{L}(\bar{u}) \approx I$ and $\bar{u} \in \pi^{\leq K}\ell^1$, an **finite** approximate zero of F.

Goal: existence of $\tilde{u}$, such that $G(\tilde{u}) = 0$.

Newton-Kantorovich Theorem

Assume there exist Y, Z_1, Z_2, and $r^* > 0$ satisfying:

(3a) $$\|AF(\bar{u})\|_{\ell^1} \leq Y,$$

(3b) $$\|I - A\mathcal{L}(\bar{u})\|_{\mathscr{B}(\ell^1)} \leq Z_1,$$

(3c) $$\|A(\mathcal{L}(u) - \mathcal{L}(\bar{u}))\|_{\mathscr{B}(\ell^1)} \leq Z_2, \quad \forall u \in B(\bar{u}, r^*).$$

(4a) $$Z_1 < 1,$$

(4b) $$2YZ_2 < (1 - Z_1)^2.$$

Then, for all r such that

(5) $$\frac{1 - Z_1 - \sqrt{(1 - Z_1)^2 - 2YZ_2}}{Z_2} \leq r < \min\left(r^*, \frac{1 - Z_1}{Z_2}\right),$$

there exists a unique steady state $\tilde{u}$ of (1) such that $\|\tilde{u} - \bar{u}\|_{\ell^1} \leq r$.

The operator A

Rewrite $\mathcal{L}(u) = \Delta\mathcal{A}(u) + \partial_x\mathcal{B}(u) + \mathcal{C}(u)$ (assume we have polynomial nonlinearities at most).
Since $\bar{u} \in \pi^{\leq K}\ell^1$ we have $\mathcal{A}(\bar{u}), \mathcal{B}(\bar{u}), \mathcal{C}(\bar{u})$ belong to $\pi^{\leq \dot{K}}\ell^1$ for a certain $\dot{K} \geq K$.

Cooking recipe for A:

- $A \in \mathscr{B}(\ell^1)$
- $\pi^{\leq N}A\pi^{\leq N} \approx \left(\pi^{\leq N}\mathcal{L}(\bar{u})\pi^{\leq N}\right)^{-1}$ in $\mathrm{Mat}_{N,N}(\mathbb{R})$,
- $A = \pi^{\leq N}A\pi^{\leq N} + a\Delta^{-1} - \pi^{\leq N}a\Delta^{-1}\pi^{\leq N}$
- with $a \approx \mathcal{A}(\bar{u})^{-1}$ in $\pi^{\leq \dot{K}}\ell^1$.

The bounds

Derived from (3a), (3b) and (3c),

$$Y = \|A\pi^{\leq \dot{K}} F(\bar{u})\|_{\ell^1}, \tag{6}$$

$$Z_1 = \max\Big(\|(I - A\mathcal{L}(\bar{u}))\pi^{\leq N+\dot{K}}\|_{\mathcal{B}(\ell^1)}, \|I_{\ell^1} - a\mathcal{A}(\bar{u})\|_{\ell^1} + \frac{1}{N}\|a\|_{\ell^1}\|\mathcal{B}(\bar{u})\|_{\ell^1} + \frac{1}{N^2}\|a\|_{\ell^1}\|\mathcal{C}(\bar{u})\|_{\ell^1} \Big), \tag{7}$$

$$Z_2 = C_{\mathcal{L}} \max(\|A\Delta\pi^{\leq N}\|, \|a\|_{\ell^1}) \tag{8}$$

Resume

For the existence of a steady state of (1) :

- Find $\bar{u} \in \pi^{\leq K}\ell^1$ such that $F(\bar{u}) \approx 0$ — sometimes the most difficult part.
- Build A and deduce Y, Z_1, Z_2 from (6), (7) and (8).
- Check (4a) and (4b) to obtain r from (5) and existence of $\tilde{u}$.

Application. See [Bre22]

$$F(u) = \Delta u^2 + u - u^2 + g,$$
$$\mathcal{L}(u)h = 2\Delta(u \star h) + h - 2u \star h,$$
$$\mathcal{A}(u) = 2u,$$
$$N = 40 \text{ and } K = 20,$$
$$Y = 1.78 \times 10^{-12},$$
$$Z_1 = 1.88 \times 10^{-3},$$
$$Z_2 = 1.64.$$

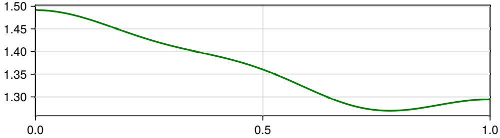
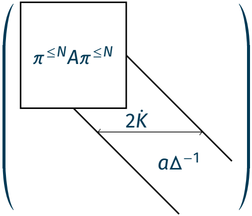


Figure: Plot of $\bar{u}$.

We conclude about the existence of $\tilde{u}$ and

$$\|\tilde{u} - \bar{u}\|_{\ell^1} \leq 1.82 \times 10^{-12} = r.$$

$A =$



Guiding idea

We want $Q = -(P\mathcal{L}(\tilde{u}) + \mathcal{L}(\tilde{u})^*P) \geq \mu$, for a certain $\mu > 0$. For a large $M \in \mathbb{N}$, $\mathcal{L}(\tilde{u})\pi^{>M} \approx \Delta\mathcal{A}(\tilde{u})\pi^{>M}$ so we can expect to build Q s.t. $Q\pi^{>M} \approx -\Delta\pi^{>M}$.

Cooking recipe for P:

- $P \in \mathscr{B}(\ell^1)$
- $\pi^{\leq N}P\pi^{\leq N}$ a solution of $X\pi^{\leq N}\mathcal{L}(\bar{u})\pi^{\leq N} + \pi^{\leq N}(\mathcal{L}(\bar{u})\pi^{\leq N})^*X = -\Delta\pi^{\leq N} + \pi^{\leq 0}$, in $\mathrm{Mat}_{N,N}(\mathbb{R})$,
- $P = \pi^{\leq N}P\pi^{\leq N} + p - \pi^{\leq N}p\pi^{\leq N}$
- with $p \in \pi^{\leq \dot{K}}\ell^1$ s.t. $q_2 := p \star \mathcal{A}(\bar{u}) + \mathcal{A}(\bar{u})^* \star p \approx -1$, in ℓ^1.

Indeed, for this P,

$$-(P\mathcal{L}(\bar{u}) + \mathcal{L}(\bar{u})^*P)\pi^{>M} = (q_2\Delta + q_1 \cdot \nabla + q_0)\pi^{>M}, \tag{9}$$

with $q_0, q_1, q_2 \in \pi^{\leq 2\dot{K}}\ell^1$

P positive

Let $\mu > 0$, let $q \in \pi^{\leq \dot{K}}\ell^1$ invertible (i.e $q > 0$).

$$\begin{aligned} q(P-\mu I)q &= q(p-\mu)q + \pi^{\leq N+\dot{K}} q(P^{\leq N} - \pi^{\leq N}(p-\mu)\pi^{\leq N}) q \pi^{\leq N+\dot{K}} \\ &= (q(p-\mu)q - I) + \pi^{>N+\dot{K}} \\ &\qquad + \underbrace{\pi^{\leq N+\dot{K}} + \pi^{\leq N+\dot{K}} q(P^{\leq N} - \pi^{\leq N}(p-\mu)\pi^{\leq N}) q \pi^{\leq N+\dot{K}}}_{:=S^{\leq N+\dot{K}}}. \end{aligned}$$

Proposition

Let $q \in \pi^{\leq \dot{K}}\ell^1$ such that $\|p - \mu - q^{-2}\|_{\ell^1} \leq \mu$.
If $S^{\leq N+\dot{K}}$ is a positive matrix then P is positive-definite operator.

Proof of Proposition

Proof.

If $S^{\leq N+\dot{K}}$ is positive then $S^{\leq N+\dot{K}} = (C^{\leq N+\dot{K}})^* C^{\leq N+\dot{K}}$, we define $C = (C^{\leq N+\dot{K}} + \pi^{>N+\dot{K}})q^{-1}$. So for all $u \in \ell^2$,

$$\begin{aligned}
\langle Pu, u\rangle_{\ell^2} - \mu\|u\|_{\ell^2}^2 &= \langle (P - \mu I - C^* C)u, u\rangle_{\ell^2} + \langle C^* Cu, u\rangle_{\ell^2} \\
&= \langle (p - \mu - q^{-2}) \star u, u\rangle_{\ell^2} + \|Cu\|_{\ell^2}^2 \\
&\geq \langle (p - \mu - q^{-2})u, u\rangle_{L^2} \\
&\geq -\|p - \mu - q^{-2}\|_{L^\infty} \|u\|_{L^2}^2 \\
&\geq -\|p - \mu - q^{-2}\|_{\ell^1} \|u\|_{\ell^2}^2
\end{aligned}$$

$\square$

Application

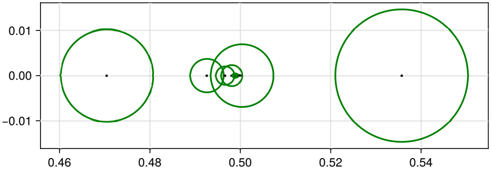
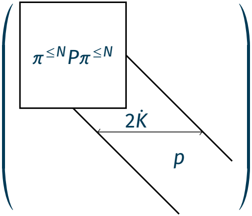


Figure: Enclosure of $\sigma(S^{\leq N+\dot{K}})$.

$P =$ 

We have $\mathcal{A}(\bar{u}) = 2\bar{u}$. We find $p \in \pi^{\leq \dot{K}} \ell^1$ s.t. $p \star \mathcal{A}(\bar{u}) + \mathcal{A}(\bar{u})^* \star p \approx -1$, with $\dot{K} = 20$.

Fix $\mu = 10^{-3}$, we find $q \in \pi^{\leq \dot{K}} \ell^1$ s.t. $\|p - \mu - q^{-2}\|_{\ell^1} = 4.94 \times 10^{-12}$

Analysis of $\sigma(Q)$

We can't deal with $Q = -(P\mathcal{L}(\tilde{u}) + \mathcal{L}(\tilde{u})^* P)$ but we have a better understanding of $\overline{Q} := -(P\mathcal{L}(\bar{u}) + \mathcal{L}(\bar{u})^* P)$.

From (9), we find μ_∞ s.t. $\sigma_{|_{\mathscr{B}(\pi^{>M}\ell^1)}}(\overline{Q}\pi^{>M}) \subset \{z \in \mathbb{C}, \mathrm{Re}(z) > \mu_\infty\}$ by studying the Gershgorin disks associated to $\overline{Q}\pi^{>M}$ — radii to bound

For $\overline{Q}\pi^{\leq M}$, we compute exactly the Gershgorin disks of the matrix.

Finally, from (8) we deduce
$\|\Delta^{-1}(Q - \overline{Q})\|_{\mathscr{B}(\ell^1)} \leq 2\|P\|_{\mathscr{B}(\ell^1)} C_{\mathcal{L}} r := C_Q$, it means that
$\sigma(Q) \subset \bigcup_{k=0}^{\infty} D(\lambda_k, C_Q k^2)$ where $\left\{\lambda_k \in \mathbb{R}, (\lambda_k) \text{ increasing} \right\} = \sigma(\overline{Q})$.

Application

We compute $q_0, q_1, q_2 \in \pi^{\leq \dot{K}} \ell^1$.

With $M = 60$, we deduce $C_Q = 3.15 \times 10^{-12}$ and $\mu_\infty = 25767$.

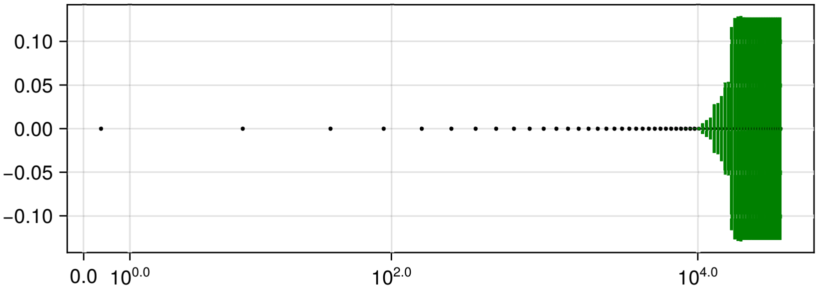


Figure: Enclosure of $\sigma(Q^{\leq M})$.

Resume

For the stability of $\tilde{u}$ solution of (1):

- Build P and check with Proposition that P is positive.
- Build $\overline{Q}$ and analyse $\sigma(\overline{Q})$ and deduce $\mu > 0$ such that $\sigma(Q) \subset \{z \in \mathbb{C}, \mathrm{Re}(z) \geq \mu\}$.
- Since Q is self-adjoint and has a compact resolvent, we have $Q \geq \mu$.

Conclusion & Outlook

We have a complete method to prove the existence of a steady state and analyse stability

We can generalize this method to n species and d dimension.

We are finishing a case of study in 2-species and 2-dimensions.

We want to find more complexe systems to see the limitations of the method.

Thank You!

[BP24] Maxime Breden and Maxime Payan. “Computer-assisted proofs for the many steady states of a chemotaxis model with local sensing”. In: *Physica D: Nonlinear Phenomena* (2024), p. 134221.

[BRE22] Maxime Breden. “Computer-assisted proofs for some nonlinear diffusion problems”. In: *Communications in Nonlinear Science and Numerical Simulation* 109 (2022), p. 106292.

[DAT70] Richard Datko. “Extending a theorem of AM Liapunov to Hilbert space”. In: *Journal of Mathematical analysis and applications* 32.3 (1970), pp. 610–616.