

Exact boundary controllability of semilinear wave equations

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with Jérôme Lemoine and Arnaud Münch



Framework and objective

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Framework. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, a bounded domain sufficiently smooth and $T > 0$. We denote

$$Q_T := \Omega \times (0, T), \quad \Sigma_T := \partial\Omega \times (0, T).$$

We consider

$$\begin{cases} \partial_{tt}y - \Delta y + f(y) = 0, & Q_T, \\ y = v|_{\Gamma_0}, & \Sigma_T, \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), & \Omega, \end{cases} \quad (*)$$

where $v \in L^2(\Sigma_T)$, $\Gamma_0 \subset \partial\Omega$ non-empty and $f \in \mathcal{C}(\mathbb{R})$ is a **non-linear function**.

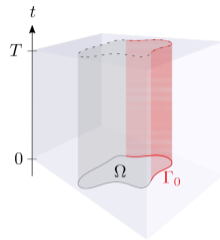


Figure – Control zone Γ_0 ,
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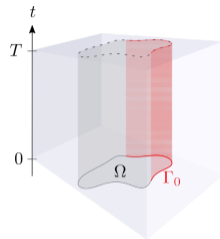


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Exact controllability problem.

Given $T > 0$, $\Gamma_0 \subset \partial\Omega$ and $(u_0, u_1), (z_0, z_1)$ in an appropriate space, find (if possible) a pair (y, v) solution of $(*)$ such that

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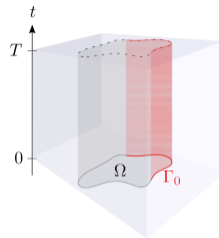


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For (u_0, u_1) and (z_0, z_1) fixed, we call **state-control pair** of $(*)$ any pair (y, v) solution of $(*)$ satisfying (1).

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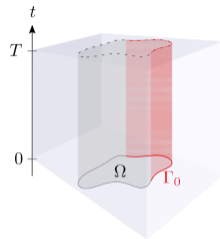


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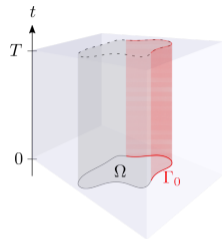


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- 1 Literature
- 2 First main result (existence of a control)
- 3 Idea of the proof
 - Linearize the system by introducing an operator Λ_S
 - Outline of the proof
 - Carleman inequality
 - Estimate of the linear optimal state-control pair
 - Existence of a fixed-point for Λ_S
- 4 Second main result (construction of a control)

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First controllability result

Controllability result [Zua91, Theorem 2.1]¹

Assume that f is **globally lipschitz** and T large enough. The system (\star) is exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$.

1. E. Zuazua, *Exact boundary controllability for the semilinear wave equation*, in *Nonlinear Partial Differential Equations and Their Applications*, Collège de France Seminar, Vol. X (Paris, 1987-1988), Pitman Res. Notes Math. Ser. 220, Longman, Harlow, UK, 1991, pp. 357-391.

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Linearization

$$\begin{cases} \partial_{tt}y - \Delta y + \frac{f(z) - f(0)}{z}y = -f(0), & Q_T, \\ y = v|_{\Gamma_0}, & \Sigma_T \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), & \Omega. \end{cases} \quad (2)$$

Fixed point operator

$$\begin{aligned} \Lambda &:= L^2(Q_T) \rightarrow L^2(Q_T) \\ z &\mapsto y \end{aligned}$$

where (y, v) is a state-control pair of (2) such that $v := \arg \min_v \|v\|_{L^2(\Sigma_T)}^2$.

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Controllability result $d = 1$, distributed control case [Zua93, Theorem 1] ²

Let T large enough. Assume that $f \in C^1(\mathbb{R})$ such that

$$\exists \beta > 0, \quad \limsup_{|r| \rightarrow \infty} \frac{|f(r)|}{|r| \ln^2 |r|} \leq \beta.$$

If β is small enough then, the system (\star) is exactly controllable at time T in $H_0^1(\Omega) \times L^2(\Omega)$.

2. E. Zuazua, *Exact controllability for semilinear wave equations in one space dimension*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 10 (1993), pp. 109-129.

Most general result known - distributed control case - Carleman setting

Let $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$ and $\varepsilon > 0$. We denote

$$\Gamma_0 := \{x \in \partial\Omega; (x - x_0) \cdot \nu(x) > 0\},$$

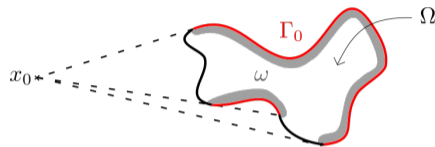
$$\mathcal{O}_\varepsilon(\Gamma_0) := \{x \in \mathbb{R}^d; \text{dist}(x, \Gamma_0) < \varepsilon\}.$$

We suppose that $\omega = \mathcal{O}_\varepsilon(\Gamma_0) \cap \Omega$ and

$$T > \max \left\{ 8 \max_{x \in \overline{\Omega}} |x - x_0|^2, 1 + 24\sqrt{d} \max_{x \in \overline{\Omega}} \{2(x - x_0) \cdot \nu(x)\} (2 + d) \right\}$$

If, moreover, the function $f \in C^1(\mathbb{R})$ satisfies

$$\exists \beta > 0, \quad \limsup_{|r| \rightarrow \infty} \frac{|f(r)|}{|r| \ln^p |r|} \leq \beta, \quad 0 \leq p < 3/2.$$



Controllability result [FLZ19, Theorem 4.5]³

Under the above conditions, if β is small enough, the system (\star) is exactly controllable in $H_0^1(\Omega) \times L^2(\Omega)$.

3. X. Fu, Q. Lü, X. Zhang, *Carleman Estimates for Second Order Partial Differential Operators and Applications : A Unified Approach*, Springer Briefs in Math., Springer, Cham, 2019.

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4 Second main result (construction of a control)

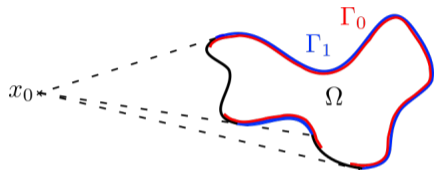
First main result (existence of a control)

For any $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$, let

- $\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}$
- $\Gamma_0 \subset \partial\Omega$ such that $\text{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0$
- $T > 2 \max_{x \in \overline{\Omega}} |x - x_0|$.

Assume that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies

$$\exists \beta > 0, \quad \limsup_{|r| \rightarrow \infty} \frac{|f(r)|}{|r| \ln^p |r|} \leq \beta, \quad 0 \leq p < 3/2.$$



Theorem [CLM24] ⁴

If β is small enough then, the system (\star) is exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$.

4. C., Lemoine, Münch, On the exact boundary controllability of semilinear wave equations. SIAM J. Control and Optimization 62(4), 1953-1976 (2024).

First main result (existence of a control)

- Extends and generalizes to any dimension [BLM23]⁵ devoted to the case $d = 1$.
- Improve [FLZ19, Theorem 4.5]⁶

	[FLZ19]	[CLM24]
Regularity of f	$C^1(\mathbb{R})$	$C^0(\mathbb{R})$
Regularity of (u_0, u_1)	$H_0^1(\Omega) \times L^2(\Omega)$	$L^2(\Omega) \times H^{-1}(\Omega)$
Lower bound of T	$\max \left\{ 8 \max_{x \in \bar{\Omega}} x - x_0 ^2, \right.$ $\left. 1 + 24\sqrt{d} \max_{x \in \bar{\Omega}} \{2(x - x_0) \cdot \nu(x)\} (2 + d) \right\}$	$2 \max_{x \in \bar{\Omega}} x - x_0 $

- The conditions on Γ_0 , T and (u_0, u_1) are those of the linear case, which are **optimal**.

5. K. Bhandari, J. Lemoine, A. Münch, *Exact boundary controllability of 1d semilinear wave equations through a constructive approach*, Math. Control Signals Systems, 1 (2023).

6. X. Fu, Q. Lü, X. Zhang, *Carleman Estimates for Second Order Partial Differential Operators and Applications : A Unified Approach*, Springer Briefs in Math., Springer, Cham, 2019.

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Zero order linearization.

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Fixed point operator.

$$\Lambda_s := \mathcal{C}(s) \subset L^\infty(0, T; L^2(\Omega)) \rightarrow \mathcal{C}(s)$$

$$z \mapsto y$$

where (y, v) is a optimal state-control pair of $(**)$ with $(z_0, z_1) = (0, 0)$ for the cost

$$\mathcal{J}_s(y, v) := s \int_{Q_T} \rho^2 |y|^2 dx dt + \int_{\Sigma_T} \eta^{-2} \Psi^{-1} \rho^2 |v|^2 dx dt,$$

involving [Carleman weight](#) $\rho(s; x, t)$ and [parameter](#) $s > 0$.

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Goal. Prove the existence of a fixed-point for Λ_s for at least one s .

- Carleman inequality (depending of s).
- Estimate of the optimal state-control pair (linear) with respect to the initial data (u_0, u_1) , the second member $f(z)$ and the parameter s .
- Verify assumptions for the Schauder theorem.

Weight function

For any $\mu \in (0, 1)$, $\lambda > 0$ and for some $M_0 > 0$ large enough so that

$$\psi(x, t) := |x - x_0|^2 - \mu \left(t - \frac{T}{2} \right) + M_0 > 0 \text{ in } \overline{Q_T},$$

$$\phi(x, t) := e^{\lambda \psi(x, t)}, \quad \rho(s; x, t) := e^{-s \phi(x, t)}, \quad \forall (x, t) \in Q_T,$$

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Carleman estimate [BdBE2013] ⁷

Under the geometric condition, there exists $s_0 > 0$, $\lambda > 0$ and $C > 0$ such that for any $s \geq s_0$,

$$\begin{aligned} s \int_{Q_T} \rho^{-2}(s) \left(|\partial_t w|^2 + |\nabla w|^2 \right) dx dt + s^3 \int_{Q_T} \rho^{-2}(s) |w|^2 dx dt + s \underbrace{\int_{\Omega} \rho^{-2}(s; x, 0) \left(|\partial_t w(x, 0)|^2 + |\nabla w(x, 0)|^2 \right) dx}_{\text{initial energy}} \\ + s^3 \int_{\Omega} \rho^{-2}(s; x, 0) |w(x, 0)|^2 dx \leq C \left(\underbrace{\int_{Q_T} \rho^{-2}(s) |\partial_{tt} w - \Delta w|^2 dx dt}_{\text{source}} + s \underbrace{\int_{\Sigma_T} \eta^2(t) \Psi(x) \rho^{-2}(s) |\partial_\nu w|^2 dx dt}_{\text{observation}} \right), \end{aligned}$$

for any $w \in \mathcal{H} := \{w \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)); \partial_{tt} w - \Delta w \in L^2(Q_T)\}$.

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A priori estimate on (y, v)

For any $r \in [0, 1] \setminus \{1/2\}$. The following estimate holds

$$\begin{aligned} & \|\rho(s)y\|_{L^2(Q_T)} + s^{-2} \|\rho(s)y\|_{L^\infty(0, T; L^2(\Omega))} + s^{-1/2} \left\| \rho(s) \eta^{-1} \Psi^{-1/2} v \right\|_{L^2(\Sigma_T)} \\ & \leq C_r \left(s^{r-3/2} \|\rho(s)f(z)\|_{L^2(0, T; H^{-r}(\Omega))} + s^{-1/2} \|\rho(0)(u_0, u_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} \right). \end{aligned} \quad (E_1)$$

Goal : Prove the existence of s and of a fixed-point for

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→ Employ the [Schauder theorem](#)

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→ Employ the [Schauder theorem](#)

- Define $\mathcal{C}(s)$ to have a **stability property** :

$$\mathcal{C}(s) := \left\{ y \in L^\infty(0, T; L^2(\Omega)); \| \rho y \|_{L^2(Q_T)} \leq s, \| \rho y \|_{L^\infty(0, T; L^2(\Omega))} \leq s^3 \right\}$$

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Recall : *A priori* estimate on y

For any $r \in [0, 1] \setminus \{1/2\}$. The following estimate holds

$$\begin{aligned}\|\rho(s)y\|_{L^2(Q_T)} + s^{-2} \|\rho(s)y\|_{L^\infty(0, T; L^2(\Omega))} \\ \leq C_r \left(s^{r-3/2} \|\rho(s)f(z)\|_{L^2(0, T; H^{-r}(\Omega))} + s^{-1/2} \|\rho(0)(u_0, u_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} \right).\end{aligned}\tag{E_1}$$

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A priori estimate on $y := \Lambda_s(z), z \in \mathcal{C}(s)$

For any $p \in (1, 3/2)$. The following estimate holds

$$\begin{aligned} & \|\rho(s)y\|_{L^2(Q_T)} + s^{-2} \|\rho(s)y\|_{L^\infty(0, T; L^2(\Omega))} \\ & \leq Cs \left(s^{-p}\alpha_2 + \beta c^p + e^{-s} \left(s^{-p-1}\alpha_1 T^{1/2} |\Omega|^{1/2} + s^{-1/2} \left(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} \right) \right) \right). \end{aligned}$$

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$$\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s)$$

$$z \mapsto y.$$

→ Employ the **Schauder theorem**

- Define $\mathcal{C}(s)$ to have a **stability property** :

$$\mathcal{C}(s) := \left\{ y \in L^\infty(0, T; L^2(\Omega)); \|\rho y\|_{L^2(Q_T)} \leq s, \|\rho y\|_{L^\infty(0, T; L^2(\Omega))} \leq s^3 \right\}$$

A priori estimate on $y := \Lambda_s(z), z \in \mathcal{C}(s)$

For any $p \in (1, 3/2)$. The following estimate holds

$$\begin{aligned} & \|\rho(s)y\|_{L^2(Q_T)} + s^{-2} \|\rho(s)y\|_{L^\infty(0, T; L^2(\Omega))} \\ & \leq Cs \left(s^{-p} \alpha_2 + \underbrace{\beta c^p}_{\Rightarrow \beta < \frac{1}{Cc^p}} + e^{-s} \left(s^{-p-1} \alpha_1 T^{1/2} |\Omega|^{1/2} + s^{-1/2} \left(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} \right) \right) \right). \end{aligned}$$

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→ For β small enough, there exists s large enough such that $\mathcal{C}(s)$ is stable under the map Λ_s .

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- **Continuity.** The map $\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s)$ is continuous for the $L^\infty(0, T; L^2(\Omega))$ -norm.
- **Relative compactness.** $\Lambda_s(\mathcal{C}(s))$ is a relatively compact subset of $\mathcal{C}(s)$ relative to the $L^\infty(0, T; L^2(\Omega))$ -norm.

Relative compactness : Any sequence $(y^n)_n$ of $\Lambda_S(\mathcal{C}(s))$ admits a subsequence $(y^{n_k})_k$ that converges in $\mathcal{C}(s)$ for the $L^\infty(0, T; L^2(\Omega))$ -norm.

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Key point - Additional regularity property on the optimal controlled pair.

Assume that $(u_0, u_1, f(z)) \in H_0^{1-r}(\Omega) \times H^{-r}(\Omega) \times L^2(0, T; H^{-r}(\Omega))$, $r \in (0, 1) \neq \{\frac{1}{2}\}$. The controlled pair (y, v) , which minimize \mathcal{J}_s , belongs to the space

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→ work with $(y^n - y^0)_n$ which is solution of linear wave equation associated with the data $(0, 0, f(z_n) - f(z_0))$.

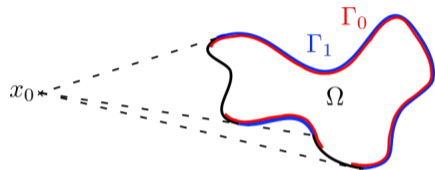
Recall of the result

For any $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$, let

- $\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}$
- $\Gamma_0 \subset \partial\Omega$ such that $\text{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0$
- $T > 2 \max_{x \in \overline{\Omega}} |x - x_0|$.

Assume that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies

$$\exists \beta > 0, \quad \limsup_{|r| \rightarrow \infty} \frac{|f(r)|}{|r| \ln^p |r|} \leq \beta, \quad 0 \leq p < 3/2.$$



Theorem [CLM24]⁸

If β is small enough then, the system (\star) is exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$.

8. C., Lemoine, Münch, On the exact boundary controllability of semilinear wave equations. SIAM J. Control and Optimization 62(4), 1953-1976 (2024).

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$$\begin{cases} \partial_{tt}y - \Delta y + f(y) = 0, & Q_T, \\ y = v|_{\Gamma_0}, & \Sigma_T, \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), & \Omega. \end{cases} \quad (*)$$

- 1 Literature
- 2 First main result (existence of a control)
- 3 Idea of the proof
 - Linearize the system by introducing an operator Λ_S
 - Outline of the proof
 - Carleman inequality
 - Estimate of the linear optimal state-control pair
 - Existence of a fixed-point for Λ_S
- 4 Second main result (construction of a control)

Assume that $f \in \mathcal{C}^1(\mathbb{R})$ and that there exists $0 \leq p < 3/2$ such that f satisfies

$$\exists \alpha, \beta > 0, \quad |f'(r)| \leq \alpha + \beta \ln^p(r), \quad \forall r \in \mathbb{R}.$$

Proposition [CLM24]⁹

For any s large enough and β small enough (s.t. $\beta < \frac{1}{Cc^p}$). Then,

$$\|\rho(s)(\Lambda_s(z_1) - \Lambda_s(z_2))\|_{L^2(Q_T)} \leq C (s^{-p}\alpha + \beta c^p) \|\rho(s)(z_1 - z_2)\|_{L^2(Q_T)}.$$

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Theorem [CLM24]⁹

For any $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists of a non trivial sequence $(y_k, v_k)_k$ that strongly converges to a state-control pair (y, v) for system (\star) . Moreover, the convergence is at least linear for the norm $\|\rho(s) \cdot\|_{L^2(Q_T)} + \|\rho(s) \cdot\|_{L^2(\Sigma_T)}$ where s is chosen sufficiently large.

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Thank you for your attention !

We consider a two-dimensional case for which $\Omega = (0, 1)^2$ and $T = 6$.

$$\begin{cases} \partial_{tt}y - \Delta y + 2.5y \left(1 + \ln^{3/2}(2 + |y|)\right) = 0, & Q_T, \\ y = v|_{\Gamma_0}, & \Sigma_T, \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (10 \sin(\pi x) \sin(\pi y), 0), & \Omega. \end{cases}$$

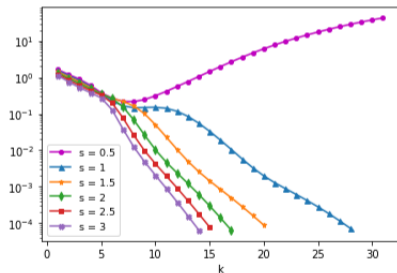
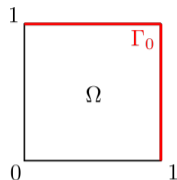


Figure – Evolution of $\frac{\|\rho^{(s)}(y_{k+1} - y_k)\|_{L^2(Q_T)}}{\|\rho^{(s)}y_k\|_{L^2(Q_T)}}$ w.r.t. k .

Continuity : Let $(z_n)_n$ be a sequence in $\mathcal{C}(s)$ such that

$$z_n \rightarrow z \in \mathcal{C}(s) \quad \text{in } L^\infty(0, T; L^2(\Omega)).$$

We denote by $y_n := \Lambda_s(z_n)$. Let us prove that $y_n \rightarrow y := \Lambda_s(z)$ in $L^\infty(0, T; L^2(\Omega))$.

$$\|\rho(s)(y_n - y_0)\|_{L^\infty(0, T; L^2(\Omega))} \leq C s^{3/2} \|\rho(s)(f(z_n) - f(z_0))\|_{L^2(0, T; H^{-1}(\Omega))}$$

- $z_n \rightarrow z$ in $L^2(Q_T)$: there exists $(z_{n_k})_k$ and $h \in L^2(Q_T)$ such that $z_{n_k} \rightarrow z$ a.e. and $|z_{n_k}| \leq h$ a.e. for any k .
- Since $f \in \mathcal{C}(\mathbb{R})$, $f(z_{n_k}) \rightarrow f(z)$ a.e.
- Moreover

$$|f(z_{n_k})| \leq \alpha_1 + |z_{n_k}| (\alpha_2 + \beta \ln^p_+ |z_{n_k}|) \leq \alpha_1 + \alpha_2 |h| + \beta |h| \ln^{3/2} |h| \leq \alpha_1 + \alpha_2 |h| + C_\varepsilon \beta |h|^{1+\varepsilon} \in L^{\frac{2}{1+\varepsilon}}(Q_T).$$

\Rightarrow For any $\varepsilon < 1$, $f(z_{n_k}) \rightarrow f(z)$ in $L^{\frac{2}{1+\varepsilon}}(Q_T)$ (in fact $f(z_n) \rightarrow f(z)$ in $L^{\frac{2}{1+\varepsilon}}(Q_T)$)

We conclude by using $L^{\frac{2}{1+\varepsilon}}(\Omega) \hookrightarrow H^{-1}(\Omega)$ if ε is small enough.