How to improve expressivity of convex ReLU neural networks?

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The ability to implement convex functions with neural networks is crucial in many applications, from learning convex regularizers for inverse problems tasks [7] to learning optimal transport maps [5, 2]. The dominant approach, widely adopted by the community, is to use Input Convex Neural Networks (ICNNs) [1]. Their main advantage is that they require only slight changes to standard neural architectures, namely an additional nonnegativity constraint on the weight matrices. While this architecture is straightforward to implement and guarantees convexity by design, ICNNs demonstrate poor expressivity when scaling up [6]. In fact, the nonnegativity constraint of ICNNs can seem somewhat arbitrary, questioning whether it is the only way to enforce convexity or if there exist many more convex neural networks that are not ICNNs. The authors in [3] provide a first answer, showing that any convex function defined over a compact domain and implemented by a ReLU neural network, *i.e.*, a piecewise affine function, can also be implemented by an ICNN. Yet, their constructive proof yields an architecture which has as many layers as affine pieces in the function, and only one neuron per layer, thus far from a practical architecture. Following this work, we study in [4] the expressivity of ICNNs for *a given architecture* (*i.e.*, set width and depth). We fully characterize convex ReLU neural networks and show that there exist convex functions implemented by a given ReLU network that are *not implementable* by any ICNN with the same architecture.

[1] B. Amos, L. Xu, J. Z. Kolter. *Input convex neural networks*, 2017.

[2] C. Bunne, S. G. Stark, G. Gut, J. S. Del Castillo, M. Levesque, K.-V. Lehmann, L. Pelkmans, A. Krause, G. Rätsch. *Learning single-cell perturbation responses using neural optimal transport.* Nature methods, **20(11)**, 1759–1768, 2023.

[3] Y. Chen, Y. Shi, B. Zhang. *Optimal Control Via Neural Networks : A Convex Approach.* In *ICLR*, 2019.

[4] A. Gagneux, M. Massias, E. Soubies, R. Gribonval. *Convexity in ReLU Neural Networks : beyond ICNNs ?*, 2025.

[5] A. Korotin, V. Egiazarian, A. Asadulaev, A. Safin, E. Burnaev. *Wasserstein-2 generative networks.* arXiv preprint arXiv :1909.13082, 2019.

[6] A. Korotin, D. Selikhanovych, E. Burnaev. *Neural optimal transport*, 2023.

[7] S. Mukherjee, S. Dittmer, Z. Shumaylov, S. Lunz, O. Öktem, C.-B. Schönlieb. *Data-driven convex regularizers for inverse problems*, 2024. doi :10.1109/ICASSP48485.2024.10447719.

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