Space-time adaptive algorithms and a posteriori estimators for mesh modification.

Théophile Boinnard,

Marco Picasso,

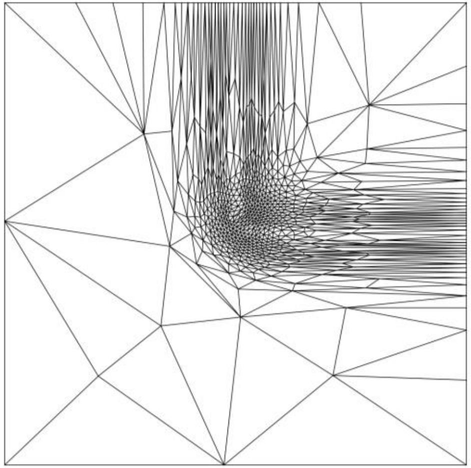
École Polytechnique Fédérale de Lausanne,

Suisse

SMAI 2025

Goal : derive and test anisotropic error estimators with mesh change

Anisotropic meshes



Kunert (2000), Formaggia and Perotto (2001, 2003),Picasso (2003), Frey, Alauzet (2005), Coupez (2011)

Time adaptivity

$$\tau^{n+1} = t^{n+1} - t^n \neq Cste$$

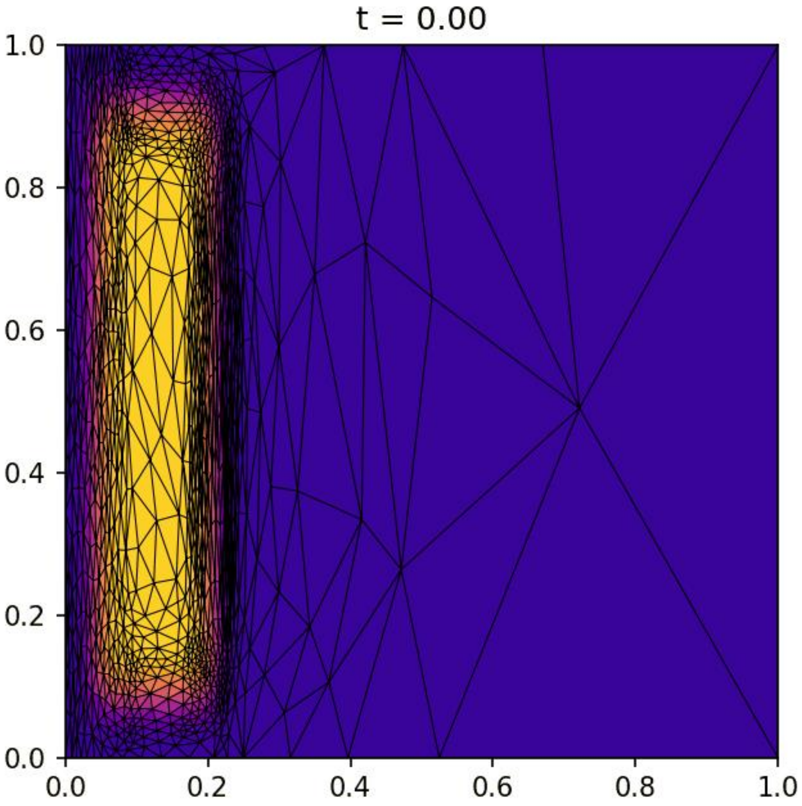
Picasso (order 1) (1998), Akrivis, Makridakis, Nochetto (order 2) (2006), Lozinski, Picasso, Prachittham (order 2) (2009)

Mesh change taken into account

$$\mathcal{T}_h^{n+1} \neq \mathcal{T}_h^n$$
$$V_h^{n+1} \neq V_h^n$$

Verfürth (2003), Bänsch, Karakatsani, Makridakis (2013), Grote, Lakkis, Santos (2024)

Example – convection-diffusion



$$\partial_t u + \beta \cdot \nabla u = \varepsilon \Delta u + f$$

$$\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Aadaptive algorithm

$$(error)^2 \sim \eta_x^2 + \eta_t^2$$

Goal : Keep the error controlled : $l.b. \leq (error)^2 \leq r.b.$

Initialize $\mathcal{T}_h^0, u_h^0, \tau^1, t = t^0$

While $t < T$:

Compute u_h^{n+1}

Compute η_x, η_t

If $\eta_x^2 \leq l.b./2$ or $\eta_x^2 \geq r.b./2$:

Adapt the mesh

If $\eta_t^2 \leq l.b./2$ or $\eta_t^2 \geq r.b./2$:

Adapt the time step τ

If $l.b. \leq \eta_x^2 + \eta_t^2 \leq r.b.$:

Move to the next time step

Problem setting – fixed mesh

Seek $u \in L^2\left(0,T; H_0^1(\Omega)\right) \cap \mathcal{C}^0\left(0,T; L^2(\Omega)\right)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega uv + \varepsilon\int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\beta\cdot\nabla u)v = \int_\Omega fv,$$
$$\forall v \in H_0^1(\Omega), a.e.\ t \in [0,T]$$

$$\varepsilon > 0$$
$$\beta \in \left(\mathcal{C}^1(\Omega)\right)^2$$
$$\nabla\cdot\beta = 0$$
$$f \in L^2(\Omega)$$

Finite elements method

$$V_h = \left\{v \in \mathcal{C}^0(\overline{\Omega}), v\Big|_K \in \mathbb{P}^k(K), \forall K \in \mathcal{T}_h\right\} \cap H_0^1(\Omega)$$

+

Euler scheme

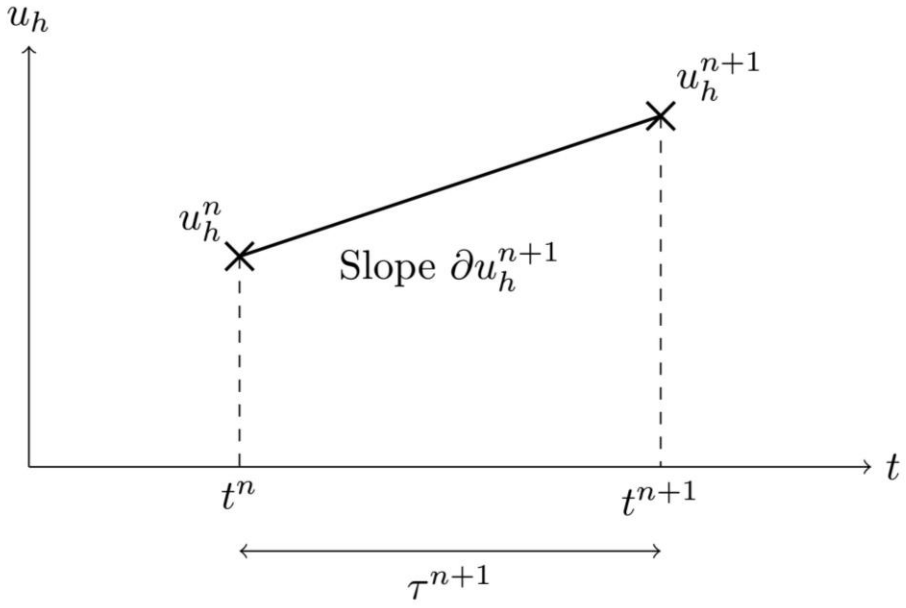
$$t^0 = 0, \dots, t^N = T,$$
$$\tau^{n+1} = t^{n+1} - t^n > 0$$

Seek $\{u_h^n\}_{n=0}^N \subset (V_h)^{N+1}$ such that

$$\int_\Omega \frac{u_h^{n+1} - u_h^n}{\tau^{n+1}} v_h + \varepsilon\int_\Omega \nabla u_h^{n+1}\cdot\nabla v_h + \int_\Omega \left(\beta\cdot\nabla u_h^{n+1}\right)v_h = \int_\Omega f^{n+1} v_h,$$
$$\forall v_h \in V_h, \forall n \in [0:N-1]$$

Time reconstruction of the solution

$$u_{h\tau}(x,t) = u_h^n(x) + (t - t^n)\partial u_h^{n+1}(x), \forall x \in \Omega, \forall t \in [t^n, t^{n+1}]$$



$$\partial u_h^{n+1} = \frac{u_h^{n+1} - u_h^n}{\tau^{n+1}}$$

$$e = u - u_{h\tau}$$

A posteriori error estimator – fixed mesh

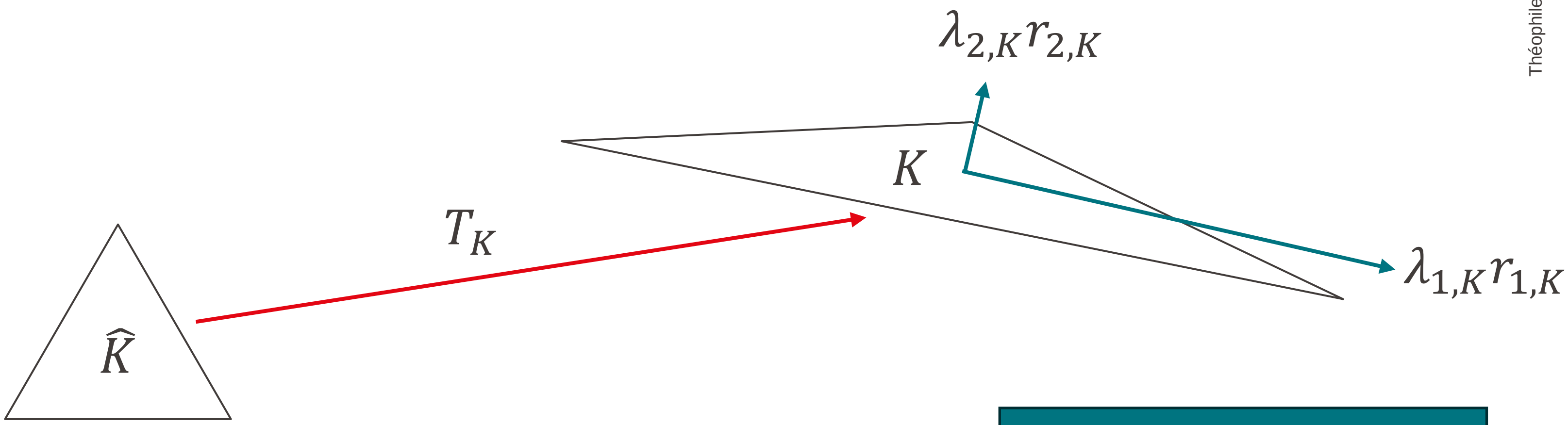
$$\frac{1}{\varepsilon}\|e(T)\|^2_{L^2(\Omega)} + \int_0^T \|\nabla e\|^2_{L^2(\Omega)} \mathrm{d}t \leq \frac{1}{\varepsilon}\|e(0)\|^2_{L^2(\Omega)} + C_x \eta_x^2 + C_t \eta_t^2$$

$$\eta_x^2 = \sum_n \sum_{K \in \mathcal{T}_h} \int_{t^n}^{t^{n+1}} \left(\left\| \frac{1}{\varepsilon}(f - \partial_t u_{h\tau} - \beta \cdot \nabla u_{h\tau}) + \Delta u_{h\tau} \right\|_{L^2(K)} + \frac{1}{2\sqrt{\lambda_{2,K}}} \|[\nabla u_{h\tau} \cdot \nu]\|_{L^2(\partial K)} \right) \omega_K(e) \mathrm{d}t$$

$$\eta_t^2 = \sum_n \int_{t^n}^{t^{n+1}} \frac{1}{\varepsilon^2} \|f - f^{n+1}\|^2_{L^2(\Omega)} \mathrm{d}t + \frac{\left(\tau^{n+1}\right)^3}{3} \left(\|\nabla \partial u_h^{n+1}\|^2_{L^2(\Omega)} + \frac{|\beta|^2_\infty}{\varepsilon^2} \|\partial u_h^{n+1}\|^2_{L^2(\Omega)} \right)$$

Anisotropic error estimator

Setting of Formaggia and Perotto - 1



$$x = T_K(\hat{x}) = M_K \hat{x} + t_K \xrightarrow{\text{SVD}} M_K = R_K^T \Lambda_K P_K$$

$$\Lambda_K = \begin{pmatrix} \lambda_{1,K} & 0 \\ 0 & \lambda_{2,K} \end{pmatrix}$$

$$R_K = \begin{pmatrix} r_{1,K}^T \\ r_{2,K}^T \end{pmatrix}$$

Anisotropic error estimator

Setting of Formaggia and Perotto - 2

Lemma: Let $R_h : H^1(\Omega) \to V_h$ be the Clément interpolant. There exists a constant $C > 0$ independent of the mesh size and aspect ratio such that, for any $v \in H_0^1(\Omega)$ and any $K \in \mathcal{T}_h$, we have:

$$\|v - R_h v\|_{L^2(K)}^2 + \lambda_{2,K} \|v - R_h v\|_{L^2(\partial K)}^2 + \lambda_{2,K}^2 \|\nabla(v - R_h v)\|_{L^2(K)}^2 \leq C\omega_K^2(v)$$

$$\begin{aligned} \omega_K^2(v) = \lambda_{1,K} r_{1,K}^T G_K(v) r_{1,K} \\ + \lambda_{2,K} r_{2,K}^T G_K(v) r_{2,K} \end{aligned}$$

$$G_K(v) = \int_{\Delta K} \begin{pmatrix} (\partial_x v)^2 & \partial_x v \partial_y v \\ \partial_x v \partial_y v & (\partial_y v)^2 \end{pmatrix}$$

Strength : indicates the contribution of each direction in the error

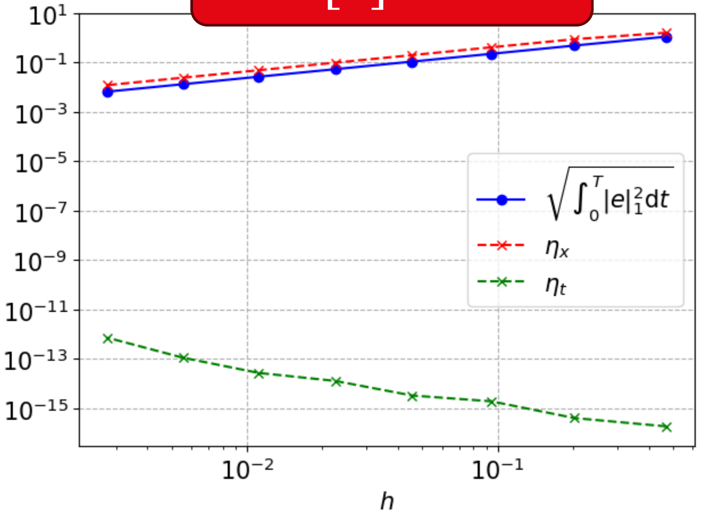
Numerical experiments – fixed mesh

- Solve the problem for a given mesh and time step.
- Two settings : the error on space dominates / the error on time dominates.

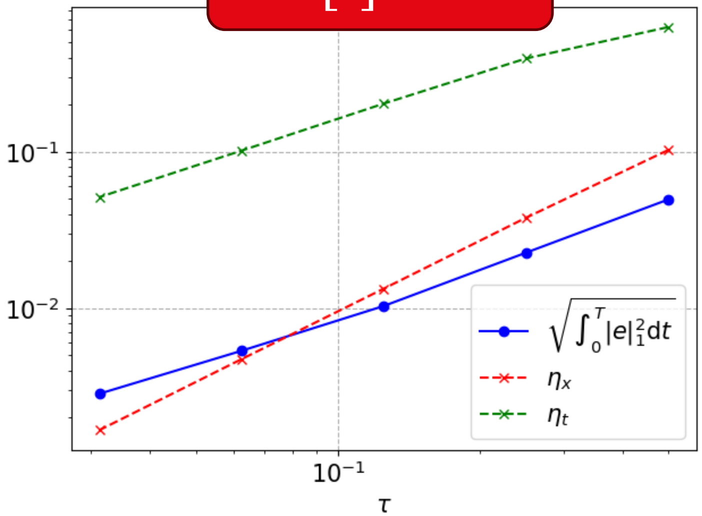
$$E = \sqrt{\int_0^T \|\nabla e\|^2_{L^2(\Omega)} \mathrm{d}t}$$

$$e.i.[x] = \eta_x / E$$
$$e.i.[t] = \eta_t / E$$

$e.i.[x] \to 1.8$



$e.i.[t] \to 20$



First issue with the change of mesh - scheme

$$\int_\Omega \frac{u_h^{n+1} - u_h^n}{\tau^{n+1}} v_h^{n+1} + \varepsilon \int_\Omega \nabla u_h^{n+1} \cdot \nabla v_h^{n+1} + \int_\Omega (\beta \cdot \nabla u_h^{n+1}) v_h^{n+1} = \int_\Omega f^{n+1} v_h^{n+1}$$

$\in V_h^n$

$\in V_h^{n+1}$

Two solutions

Project u_h^n onto V_h^{n+1}

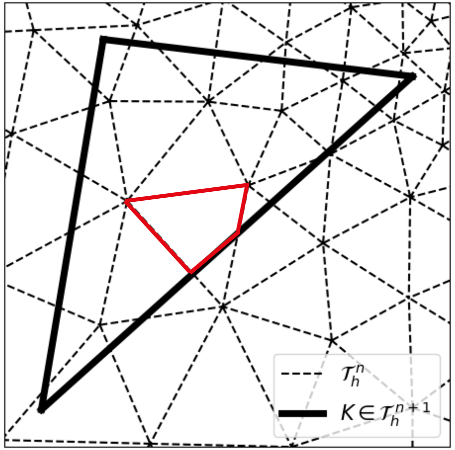
Compute exactly $\int_\Omega u_h^n v_h^{n+1}$

Lagrange interpolant
$u_h^n \leftarrow r_h^{n+1} u_h^n$

L^2 –stability

L^2 –projection
$u_h^n \leftarrow \Pi_h^{n+1} u_h^n$
Where
$$\int_\Omega u_h^n v_h^{n+1} = \int_\Omega \Pi_h^{n+1} u_h^n v_h^{n+1}$$

Second issue with the change of mesh - estimators



$$u_{h\tau} = u_h^n + \frac{t - t^n}{\tau^{n+1}}\left(u_h^{n+1} - u_h^n\right)$$

$\in V_h^n$ $\quad$ $\in V_h^{n+1}$ $\quad$ $\in V_h^n$

Efficient algorithm to compute mesh intersection : M.J. Gander and C. Japhet (2008)

$$\eta_x^2 = \sum_n \sum_{K \in \mathcal{T}_h} \int_{t^n}^{t^{n+1}} \left(\left\| \frac{1}{\varepsilon}(f - \partial_t u_{h\tau} - \beta \cdot \nabla u_{h\tau}) + \Delta u_{h\tau} \right\|_{L^2(K)} + \frac{1}{2\sqrt{\lambda_{2,K}}} \| [\nabla u_{h\tau} \cdot \nu] \|_{L^2(\partial K)} \right) \omega_K(e) \mathrm{d}t$$

$$\eta_x^2 = \sum_n \sum_{K \in \mathcal{T}_h} \int_{t^n}^{t^{n+1}} \left(\sum_{P \subset K} \left\| \frac{1}{\varepsilon}(f - \partial_t u_{h\tau} - \beta \cdot \nabla u_{h\tau}) + \Delta u_{h\tau} \right\|_{L^2(P)} + \frac{1}{2\sqrt{\lambda_{2,K}}} \| [\nabla u_{h\tau} \cdot \nu] \|_{L^2(\partial P)} \right) \omega_K(e) \mathrm{d}t$$

Second issue with the change of mesh - estimators

Usually,

$$\frac{u_h^{n+1} - u_h^n}{\tau^{n+1}} \leftarrow \frac{u_h^{n+1} - \Pi_h^{n+1} u_h^n}{\tau^{n+1}}$$

Here, we compute exactly

$$\int_\Omega u_h^n u_h^{n+1}$$

$$\eta_t^2 = \sum_n \int_{t^n}^{t^{n+1}} \frac{1}{\varepsilon^2} \| f - f^{n+1} \|_{L^2(\Omega)}^2 \mathrm{d}t + \frac{\left(\tau^{n+1}\right)^3}{3} \left(\| \nabla \partial u_h^{n+1} \|_{L^2(\Omega)}^2 + \frac{|\beta|_\infty^2}{\varepsilon^2} \| \partial u_h^{n+1} \|_{L^2(\Omega)}^2 \right)$$

$$\left(\tau^{n+1}\right)^2 \| \partial u_h^{n+1} \|_{L^2(\Omega)}^2 = \| u_h^{n+1} \|_{L^2(\Omega)}^2 + \| u_h^n \|_{L^2(\Omega)}^2 - 2 \int_\Omega u_h^n u_h^{n+1}$$

Two strategies

Given $\{\mathcal{T}_h^n\}_n, \{u_h^n\}_n$

Replace u_h^n by its projection

- Easy to implement
- Leverage capabilities of most FEM libraries

TOL	$e.i.[x]$	$e.i.[t]$	$e.i.$
0.5	1.11	1.10	1.57
0.25	1.17	0.94	1.50
0.125	1.18	0.93	1.50
0.0625	1.17	1.23	1.70
0.03125	1.17	1.24	1.70

Use the new estimators

- Upper bound guaranteed by the theorem

TOL	$e.i.[x]$	$e.i.[t]$	$e.i.$
0.5	0.99	1.10	1.48
0.25	1.03	0.94	1.39
0.125	1.03	0.93	1.39
0.0625	1.02	1.23	1.60
0.03125	1.02	1.24	1.60

Conclusion

- Derived a residual based anisotropic error estimator with adapted, possibly non-nested meshes.
- Verification of the estimators on adapted meshes.

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Post-processing

Issue :
$\omega_K(e)$ requires $\nabla e = \nabla u - \nabla u_h$

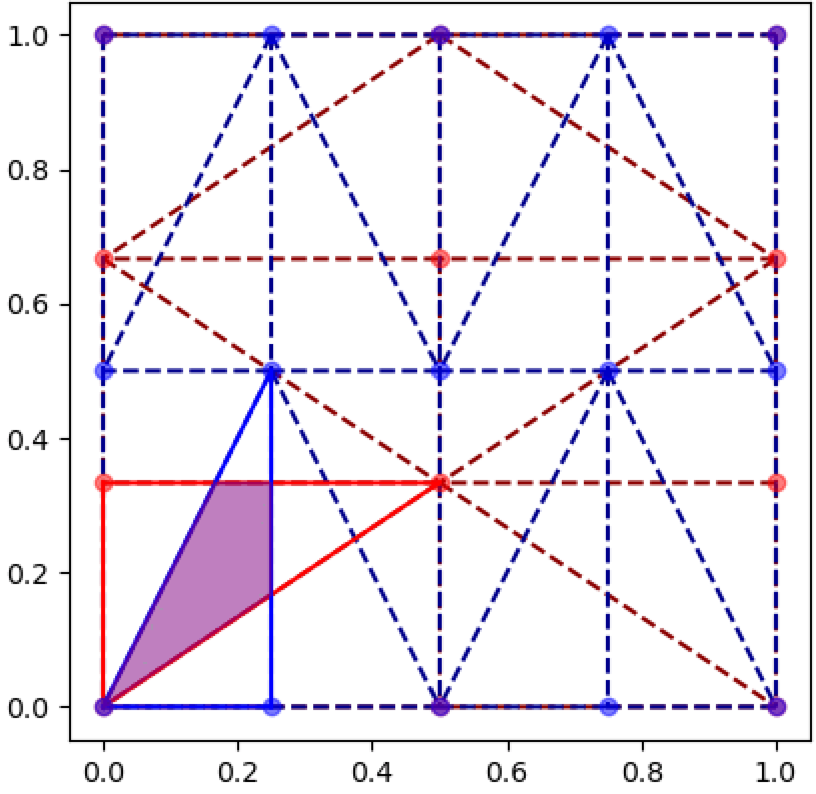
Solution :
Estimate ∇u as $G_h u_h$ (better gradient), such that
$$\underbrace{\|\nabla u - \nabla u_h\|}_{\sim h^k} \leq \underbrace{\|\nabla u - G_h u_h\|}_{\sim h^{k+1}} + \underbrace{\|G_h u_h - \nabla u_h\|}_{\sim h^k}$$

Idea : project ∇u_h back to V_h (space of u_h). I use both Zienkiewicz-Zhu ($\mathbb{P}^1$) and Naga-Zhang ($\mathbb{P}^1 - \mathbb{P}^2$)

Computation of interface mass matrix

Aim : Compute $M_{ij}^{ab} = \int_\Omega \phi_i^a \phi_j^b$

- Start from two intersecting triangles $X \in \mathcal{T}_a$, $Y \in \mathcal{T}_b$.
- Compute the intersection P, add the mortar contribution $\int_P \phi_i^a \phi_j^b$ to M_{ij}^{ab}.
- Then, check the neighbors of Y that intersect with X. Compute the contributions.
- Once there are no neighbors of Y intersecting with X, continue with the neighbors of X.



Issues with Crank-Nicolson – 1

$$\int_\Omega \frac{u_h^{n+1} - u_h^n}{\tau^{n+1}} v_h^{n+1} + \varepsilon \int_\Omega \nabla\left(\frac{u_h^{n+1} + u_h^n}{2}\right) \cdot \nabla v_h^{n+1} + \int_\Omega \beta \cdot \nabla\left(\frac{u_h^{n+1} + u_h^n}{2}\right) v_h^{n+1} = \int_\Omega f^{n+1/2} v_h^{n+1}$$

$$\int_\Omega \frac{u_h^{n+1} - \Pi_h^{n+1} u_h^n}{\tau^{n+1}} v_h^{n+1} + \varepsilon \int_\Omega \nabla\left(\frac{u_h^{n+1} + \Pi_h^{n+1} u_h^n}{2}\right) \cdot \nabla v_h^{n+1} + \int_\Omega \beta \cdot \nabla\left(\frac{u_h^{n+1} + \Pi_h^{n+1} u_h^n}{2}\right) v_h^{n+1} = \int_\Omega f^{n+1/2} v_h^{n+1}$$

L^2 –stable, but estimators with $\frac{1}{\varepsilon}\left\|u_h^n - \Pi_h^{n+1} u_h^n\right\|$ unbounded as $\varepsilon \to 0$

Issues with Crank-Nicolson - 2

ODE : $\frac{\mathrm{d}}{\mathrm{d}t}u + Au = 0$

CN : $\frac{u^{n+1}-u^n}{\tau} + A\frac{u^{n+1}+u^n}{2} = 0 \; (n+1)$

$$\frac{u^n - u^{n-1}}{\tau} + A\frac{u^n - u^{n-1}}{2} = 0 \; (n)$$

$$\tau\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + A\frac{u^{n+1} - u^{n-1}}{2} = 0$$

Finite elements :

$$\int_\Omega \frac{u_h^{n+1} - u_h^n}{\tau} v_h^{n+1} + \varepsilon \int_\Omega \nabla\left(\frac{u_h^{n+1} + u_h^n}{2}\right) \cdot \nabla v_h^{n+1} = 0$$

$$\int_\Omega \frac{u_h^n - u_h^{n-1}}{\tau} v_h^n + \varepsilon \int_\Omega \nabla\left(\frac{u_h^n + u_h^{n-1}}{2}\right) \cdot \nabla v_h^n = 0$$

Difference $v_h^{n+1} - v_h^n$?