Strong propagation of chaos for systems of interacting particles with nearly stable jumps

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α-stable r.v.s and their domains of attraction

- A r.v. X is **strictly stable of index $\alpha \in (0,2]$** if, **for all $n \geq 1$**,

$$\frac{1}{\sigma n^{1/\alpha}} \sum_{i=1}^{n} X_i \stackrel{d}{=} X$$

where $(X_i)_{i=1}^n$ are i.i.d. copies of X, $\sigma > 0$.

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- A r.v. U belongs to the normal **domain of attraction** of a stable r.v. X of index α if, taking $(U_i)_{i\geq 1}$ i.i.d. $\sim U$ and for suitably chosen constants $(a_n)_{n\geq 1}$,

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We will take:

- $\alpha \in (0,2) \setminus \{1\}$;
- ν **nearly stable** of index α, **centered if $\alpha > 1$**.

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N particles with positions $X^{N,i} \in \mathbb{R}$.

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$$X_t^{N,i} = X_0^i + \int_0^t \boldsymbol{b}(X_s^{N,i}, \boldsymbol{\mu_s^N})ds + \int_{[0,t]\times\mathbb{R}_+} \boldsymbol{\psi}(X_{s-}^{N,i}, \boldsymbol{\mu_{s-}^N})\mathbb{1}_{\{z \le \boldsymbol{f(X_{s-}^{N,i})}\}}\bar{\pi}^i(ds, dz)$$
$$+ \sum_{\boldsymbol{j \neq i}} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} \frac{\boldsymbol{u}}{\boldsymbol{N^{1/\alpha}}} \mathbb{1}_{\{z \le \boldsymbol{f(X_{s-}^{N,j})}\}}\pi^j(ds, dz, du) \qquad i = 1, \ldots, N \tag{1}$$

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We can prove **strong existence and uniqueness** for the N-particle system.

Limit system

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We can prove **strong existence and uniqueness for the limit system**.

Main result

Theorem (Löcherbach, Loukianova, M. - 2025)

*Under appropriate assumptions, for any $N \geq 1$, **we can construct a strictly α-stable process $S^{N,\alpha}$**, on an extension of the original probability space, independent of the initial conditions $(X_0^i)_i$ and of the $(\bar{\pi}^i)_i$, such that the following holds. If $(\bar{X}^i)_i$ is the solution of the limit system driven by $S^{N,\alpha}$ and $(\bar{\pi}^i)_i$, and $T_K^N \coloneqq \inf\{t \geq 0 : |\Delta S_t^{N,\alpha}| > K\}$, for any $K > 0$, $t \geq 0$ and $i = 1, \dots, N$,*

$$\mathbb{E}[\mathbb{1}_{\{t<T_K^N\}}|X_t^{N,i} - \bar{X}_t^i| \wedge |X_t^{N,i} - \bar{X}_t^i|^{\alpha-}] \leq C_t r(N) \qquad \alpha < 1$$

$$\mathbb{E}[\mathbb{1}_{\{t<T_K^N\}}|X_t^{N,i} - \bar{X}_t^i|] \leq \tilde{C}_t \tilde{r}(N) \qquad \alpha > 1$$

where $C_t, \tilde{C}_t$ are constants and r, $\tilde{r}$ explicit rates, $\alpha_- < \alpha$.

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Corollary (Löcherbach, Loukianova, M. - 2025)

Under the previous assumptions,

$$\lim_{N\to+\infty} W_{d_{\alpha\_}}(\mathcal{L}(X_t^{N,i}), \mathcal{L}(\bar{X}_t^i)) = 0 \qquad \alpha < 1$$
$$\lim_{N\to+\infty} W_1(\mathcal{L}(X_t^{N,i}), \mathcal{L}(\bar{X}_t^i)) = 0 \qquad \alpha > 1,$$

so we have ***weak convergence of $\mathcal{L}(X_t^{N,i})$ to $\mathcal{L}(\bar{X}_t^i)$*** *together with convergence of the first moments*[a].

[a] $W_{d_{\alpha\_}}(\nu_1, \nu_2) := \inf_{\pi\in\Pi(\nu_1,\nu_2)} \int_{\mathbb{R}^2} d_{\alpha\_}(x,y)\pi(dx,dy)$, $d_{\alpha\_}(x,y) := |x-y| \wedge |x-y|^{\alpha -}$.

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Understanding how S^α arises is the key to our convergence proof.

Heuristics: how we obtain the limit system

$$A_t^N := \sum_{j=1}^{N} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} \frac{u}{N^{1/\alpha}} \mathbb{1}_{\{z \le f(X_{s-}^{N,j})\}} \pi^j(ds, dz, du).$$

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If $\boldsymbol{f(x) \equiv \lambda}$ **on** $\boldsymbol{[0,t]}$, then

$$A_t^N = \left(\frac{1}{N}\right)^{1/\alpha} \sum_{k=1}^{P^N_{0,t}} U_k$$

$(U_k)_{k\geq 1}$ i.i.d. $\sim \nu$

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LLN for $P_{0,t}^N$ yields

$$A_t^N \overset{N\to\infty}{\sim} \left(\frac{N\lambda t}{N}\right)^{1/\alpha} S_1^\alpha \overset{d}{=} \lambda^{1/\alpha} S_t^\alpha = \int_{]0,t]} \lambda^{1/\alpha} dS_s^\alpha$$

where S_t^α is the increment of a stable process during time t (**self-similarity**).

Non constant rate: time discretization

If ***f* not constant**: $[k\delta, (k+1)\delta[, \forall\, k \geq 0$.

[1] See **Chen, Nourdin, Xu 2021 and Chen, Nourdin, Xu, Yang, Zhang 2022** for quantitative stable CLT.

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If **f not constant**: $[k\delta, (k+1)\delta[, \forall\, k \geq 0$.

On each interval $[k\delta, (k+1)\delta[$:

- We **freeze jumps rates** $f(X^{N,i}_{k\delta})$ (total main jump rate in k-th interval is $\sum_{i=1}^{N} f(X^{N,i}_{k\delta})$).
- We apply the approximations seen above[1] to obtain

$$\left(\frac{\textbf{total jump rate } \times \delta}{N}\right)^{1/\alpha} S_1^{\alpha,k} = \left(\frac{\sum_{j=1}^{N} f(X^{N,j}_{k\delta})}{N}\right)^{1/\alpha} \delta^{1/\alpha} S_1^{\alpha,k}.$$

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The contribution of all intervals should give

$$\sum_k \left(\frac{\sum_{i=1}^{N} f(X^{N,i}_{k\delta})}{N}\right)^{1/\alpha} \delta^{1/\alpha} S_1^{\alpha,k} \stackrel{d}{=} \sum_k (\mu^N_{k\delta}(f))^{1/\alpha} S_\delta^{\alpha,k} \xrightarrow{\delta \to 0} \int_0^t (\bar{\mu}_{s-}(f))^{1/\alpha} dS_s^\alpha.$$

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Another key step

$$\bar{X}_t^i = \bar{X}_0^i + \int_0^t b(\bar{X}_s^i, \bar{\mu}_s^N) ds + \int_{[0,t]\times\mathbb{R}_+} \psi(\bar{X}_{s-}^i, \bar{\mu}_{s-}^N) \mathbb{1}_{\{z \le f(\bar{X}_{s-}^i)\}} \bar{\pi}^i(ds, dz)$$
$$+ \int_{[0,t]} (\bar{\mu}_{s-}^N(f))^{1/\alpha} dS_s^{N,\alpha} + R_t^N$$

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Proposition (Löcherbach, Loukianova, M. - 2025)

Under appropriate assumptions, for all $N \ge 1$,

$$\mathbb{E}[\mathbb{1}_{\{t \le T_K^N\}} |R_t^N|] \le C_t \frac{K^{1-\alpha}}{1-\alpha} w(N) \qquad \alpha < 1$$
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where C_t, $\tilde{C}_t$ are positive constants and $w(N,\delta)$, $\tilde{w}(N,\delta)$ explicit rates, and $\alpha_+ \in (\alpha, 2)$.

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It is sufficient to control $W_{r(\alpha)}(\bar{\mu}_t^N, \bar{\mu}_t)$ with $r(\alpha)$ order depending on α (**Fournier, Guillin 2015**).

Norms

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For errors related to **collateral jump term and finite system**:

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Finite and limit systems have different natures.

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The jumps of $S^{N,\alpha}$ are represented by a PRM with **intensity $ds\nu^{\alpha}(dz)$**,

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In particular, for all $K > 0$,

$$\int_{B_K^c} |z|^q \nu^\alpha(dz) < +\infty \iff q < \alpha \qquad \text{and} \qquad \int_{B_K} |z|^q \nu^\alpha(dz) < +\infty \iff q > \alpha.$$

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Boundedness and Lipschitz properties of b, f, ψ allow to switch.

Work in progress and future directions

- $(\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^{N,i})_{t\geq 0}})_{N\geq 1} \xrightarrow{w} \bar{\mu} := \mathcal{L}(\bar{X}^1|S^\alpha)$ (on the **path space**).

- **Long-time** behavior of both the finite and the limit systems.

- Study of a **spatially structured** version of the model, where the interactions depend on the particle positions in $\mathbb{R}^d$.

Thank you!

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