We present a posteriori estimates for finite element approximations of nonlinear elliptic problems satisfying Lipschitz-continuity and strong-monotonicity properties. These estimates include, and build on, any iterative linearization method that satisfies a few clearly identified assumptions; this encompasses the Picard, Newton, and Zarantonello linearizations. The estimates give a guaranteed upper bound on an augmented energy difference (reliability with constant one), as well as a lower bound (efficiency up to a generic constant). We prove that for the Zarantonello linearization, this generic constant only depends on the space dimension, the mesh shape regularity, and possibly the approximation polynomial degree in four or more space dimensions, making the estimates robust with respect to the strength of the nonlinearity. For the other linearizations, there is only a computable dependence on the local variation of the linearization operators. We also derive similar estimates for the energy difference that depend locally on the nonlinearity and improve the usual bound. Numerical experiments illustrate and validate the theoretical results, for both smooth and singular solutions.