Control of collision orbits

Riccardo Daluiso[1], Jean-Baptiste Caillau, Alain Albouy

[1]Université Côte d'Azur

June 5, 2025

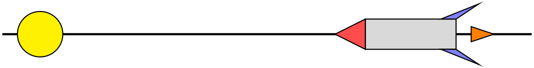


Table of Contents

Table of Contents

What does *regularization* mean

When approaching a binary collision in the n-body problem (here $n = 2$), the gravitational acceleration and the velocity of the colliding bodies tend to infinity.

The Levi-Civita regularization (1904) transforms this singular motion into a smooth motion where all the variables remain finite, and which continues after the collision. This transformation couples the change of coordinates given by the singular map $z \mapsto z^2$ with a change of a time.

SUR LA RÉSOLUTION QUALITATIVE DU PROBLÈME RESTREINT

DES TROIS CORPS

PAR

T. LEVI-CIVITA

À PADOUE.

Dans le problème des trois corps (points matériels, qui s'attirent suivant la loi de NEWTON) les forces et par conséquent les équations différentielles du mouvement se comportent d'une façon analytique régulière tant que les positions des trois points restent distinctes.

D'après cela il est presque évident qu'il ne peut y avoir autre raison de singularité pour le mouvement en dehors de la circonstance que deux des trois corps (ou tous les trois) se rapprochent indéfiniment.

Plus précisément M. PAINLEVÉ[1] a démontré qu'à partir de conditions initiales données des singularités peuvent se présenter alors seulement qu'une au moins des distances mutuelles tend vers zéro pour t convergent vers une valeur finie t_1.

Quoi qu'il en soit, les résultats récents de M. MITTAG-LEFFLER sur les représentations des branches monogènes des fonctions analytiques permettent d'affirmer que:

Dans le problème des trois corps les coordonnées sont exprimables *en tout cas et pendant toute la durée du mouvement* par des séries jouissant des propriétés fondamentales des séries de TAYLOR.

Soit en effet x une quelconque de ces coordonnées. D'après la conclusion de M. PAINLEVÉ, rappelée tout à l'heure, la fonction $x(t)$ reste

[1] Voir ses »Leçons etc., professées à Stockholm», chez A. Hermann, Paris 1897, p. 583.

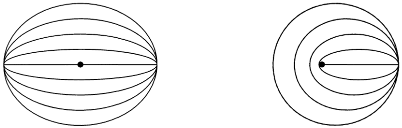
Acta mathematica. 30. Imprimé le 17 février 1906. 39

Proposition

Suppose the motion of a point in the complex plane is given by $z(\tau)$ and satisfies Hooke's law $z'' = -Cz$. Then a point following the trajectory $\omega(t(\tau)) = [z(\tau)]^2$, where $dt = |\omega| d\tau = |z|^2 d\tau$, moves according to Newton's law

$$\frac{d^2\omega}{dt^2} = -\tilde{C}\frac{\omega}{|\omega|^3},$$

where $\tilde{C} = 2\left(|z'(0)|^2 + C|z(0)|^2\right)$



Effect of the transformation $z \to z^2$ on the Hooke ellipsis

Table of Contents

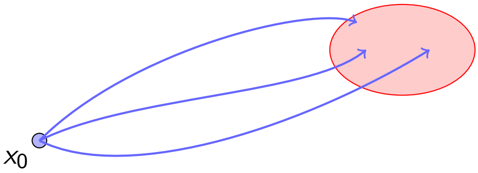
Setting

A controlled dynamical system is a smooth family of vector fields $f : M \times U \to TM$ where M is a smooth $n-$dimensional manifold, and $U \subset \mathbb{R}^m$ is the set of admissible controls.
First question: is some final state x_f accessible from some initial state x_0, i.e. does the system

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad u(t) \in U$$

$$x(0) = x_0, \qquad x(t_f) = x_f$$

have a solution for some admissible control? The system is said to be *controllable* if the answer is positive for all possible initial and final states $x_0, x_f \in M$.



Controllability

Theorem

Let

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$$

a controlled-affine system on a connected manifold, with $u = (u_1, \ldots, u_m) \in U$. If

- f_0 is periodic
- The convex enveloppe of U is a neighbourhood of the origin
- $Lie_x \{f_0, \ldots, f_m\} = T_x M$, $x \in M$

then the system is controllable.

Pontryagin maximum Principle (PMP)

Let $\dot{x} = f(x,u), \quad \int_0^{t_f} L(x,u) \to \min$, defined on $M = \mathbb{R}^n$.

Theorem (Pontryagin et al.)

iIf u is an optimal control on $[0, t_f]$, t_f not fixed, with response x, then there exists an absolutely continuous covector function p valued in $(\mathbb{R}^n)^*$ and $p^0 \leq 0$, not both zero, such that, with

$$H(x, p, p^0, u) = \langle p, f(x,u) \rangle + p^0 L(x,u),$$

almost everywhere in $[0, t_f]$,

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, p^0, u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p, p^0, u)$$

$$H(x, p, p^0, u) = \max_{v \in U} H(x, p, p^0, v).$$

Morever, one has that $H = 0$. a.e. in $[0, t_f]$.

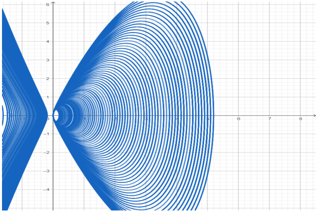
Table of Contents

PMP for the 1-dimensional Kepler problem

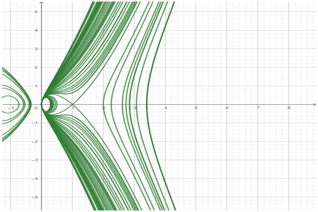
$$\ddot{q} = -\frac{q}{|q|^3} + u$$ $\boxed{\text{Controlled-Kepler Problem}}$

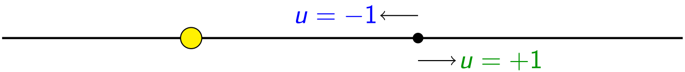
The equations arising from the Hamiltonian of the PMP restricted to collision orbits are $\ddot{q} = -q^{-2} \pm 1 \quad q \in \mathbb{R}_{>0}$, which are integrable in the new time τ, where $d\tau = 1/q dt$ in terms of the Weierstrass $\wp-$function and so well-defined at collision.

control $u = -1$



control $u = +1$





To really understand the behavior of the control system, we need to regularize the whole system - and not just the extremals arising from the *PMP*.

The Levi-Civita regularization written in the real plane as $(q_1, q_2) = \left(\xi_1^2 - \xi_2^2,\, 2\xi_1\xi_2\right)$ together with the change of time $d\tau = 1/|\xi|^2 dt$ can be extended to the controlled system

$$\ddot{q} = -\frac{q}{|q|^3} + u$$

$q \in \mathbb{R}^2, |u| \leq 1$ by giving the controlled-affined system

$$x' = f_0(x) + u_1 f_1(x) + u_2 f_2(x)$$

where $x = (\xi_1, \xi_2, \eta_1, \eta_2, h) \in \mathbb{R}^5$, and f_0, f_1, f_2 real-analytical vector fields defined on the 4−dimensional manifold

$$M = \{2|\eta|^2 + (-h)|\xi|^2 = 1\} \subset \mathbb{R}^5$$

$h := \frac{1}{2}|\dot{q}|^2 - \frac{1}{|q|}$ be the energy of the non-controlled system.

Thanks to this change of - dependent and independent - variables, we can apply the thorem for the controllability to conclude that

Theorem

The planar controlled Kepler problem is everywhere controllable, whenever $h < 0$, where $h(t) = \frac{1}{2}|\dot{q}|^2 - \frac{1}{|q|}$ is the varying energy of the non-controlled system.

Proof: By calculation, it turns out that $\mathrm{Lie}_x \{f_0, f_1, f_2\} = T_x M$, $\forall x \in M$ and the vector field f_0 is periodic whenever $h < 0$.

Existence of optimal solutions

The application of the *PMP* requires verifying the existence of an optimal solution of the regularized system. The existence of an optimal couple trajectory - control (x^*, u^*) which solves the regularized problem

$$x' = f_0(x) + u_1 f_1(x) + u_2 f_2(x), \qquad t_f = \int_0^{\tau_f} |\xi|^2 d\tau \to \min$$

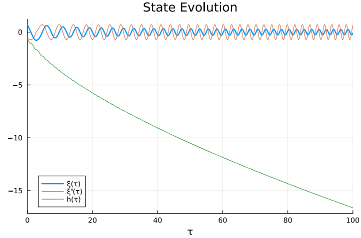
is guaranteed if a blow-up in finite time does not occur for **every** choice of admissible controls. But also by restricting to the one dimensional case, the Weierstrass $\wp-$function has poles!

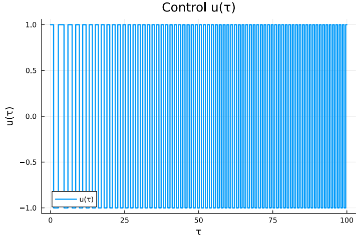
Conjecture

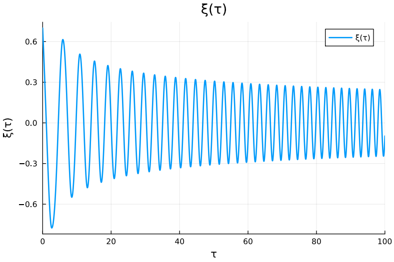
The existence of an optimal solution for the regularized Kepler problem is guaranteed only on the subset of M defined by $h < 0$

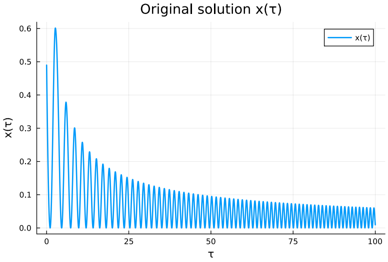
Landing on earth

We force the control so that $h'(\tau) \to \min$, i.e. $u = -\frac{\xi\xi'}{|\xi\xi'|}$



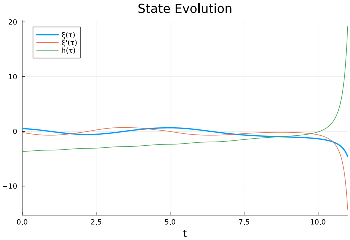


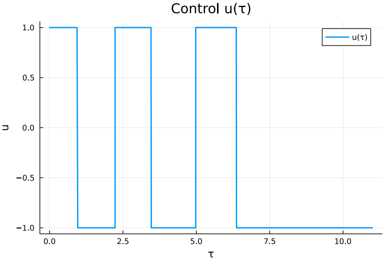


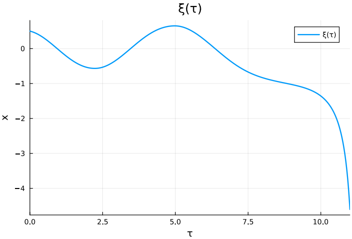


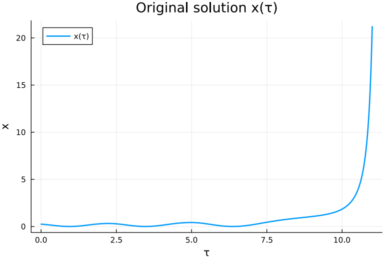
Using the 'black hole' to blow-up the spacecraft

We force the control so that $h'(\tau) \to \max$, i.e. $u = \frac{\xi\xi'}{|\xi\xi'|}\ldots$ blow-up!









What we are working on

- Bound the system in the region $h < 0$, so that existence of optimal solutions is guaranteed
- Solve the equations of the *PMP* in the regular setting
- compute the feedback u (when we need to switch?)
- Extend the results to the three-dimensional case

thank you!