Phase field approximation for Plateau's problem

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Introduction of the problem

Plateau's problem

Definition (Plateau's problem)

Finding a set that minimizes its area and spans a given boundary.

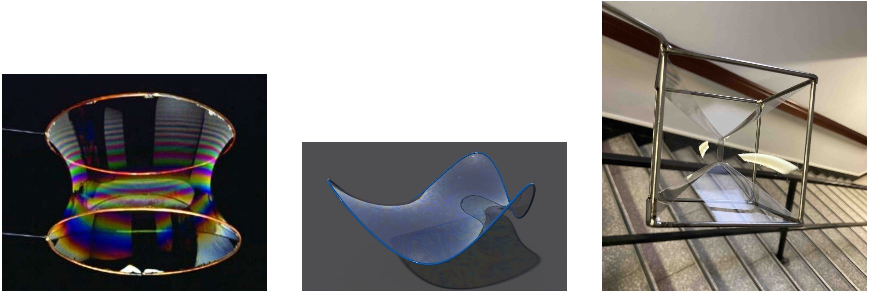

Figure: Application examples : shape of soap films

Two major approaches

- **H. Federer** (1920-2010) and **W. H. Fleming** (1928-2023): use oriented currents to solve Plateau's oriented problem.
  - ☛ **Variational approximation of size-mass energies for k-dimensional currents.**, A. Chambolle, L.A.D. Ferrari, B. Merlet, *ESAIM: Control, Optimisation and Calculus of Variations, 2019, 25 (2019).*

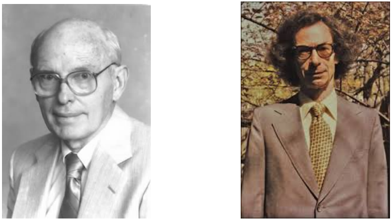

Figure: Fleming and Federer

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 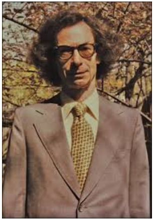

Figure: Fleming and Federer

- **E.R. Reifenberg** (1928-1964): uses Čech homology to define surface spanning a boundary.

Phase field approximation

Steiner's problem

Definition (Steiner's problem)

Finding the connected set K containing the points $a_1, ..., a_n$ and minimizing the length $\mathcal{H}^1(K)$.

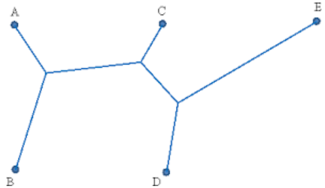

Figure: Steiner's problem with 5 points (N=4)

Remark: Steiner's problem can be seen as Plateau's problem in lower dimension. We connect points instead of curves and minimize length instead of area.

Phase field approximation

Approximation energy:

$$F_\varepsilon(u) := \varepsilon \int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (1-u)^2 dx + \frac{1}{c_\varepsilon}\sum_{i=1}^{N} d_u(a_0, a_i),$$

where the **geodesic distance** between a_0 and a_i is

$$d_u(a_0, a_i) = \inf_{\gamma : a_0 \to a_i} \int_\gamma |u|^2 d\mathcal{H}^1.$$

Ambrosio–Tortorelli energy controls the length.

The penalisation by geodesics insures the connectivity.

☛ **Approximation of length minimization problems among compact connected sets**, M. Bonnivard, A. Lemenant, and F. Santambrogio, *SIAM Journal on Mathematical Analysis, 47(2), 1489-1529 (2015).*

Definition (Geodesic distance)

The geodesic distance between two closed curves, γ_1 and γ_2, is defined by

$$d_u(\gamma_1, \gamma_2) := \inf_{\ell : \gamma_1 \longrightarrow \gamma_2} \int_{S_\ell} |u|^2 d\mathcal{H}^2,$$

where, $\ell : \gamma_1 \longrightarrow \gamma_2$ means that ℓ is a smooth curve in the space of closed curves connecting γ_1 and γ_2, and S_ℓ is the image of this curve ℓ.

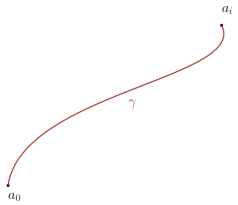

Figure: geodesic connecting points

Geodesic distance between closed curves

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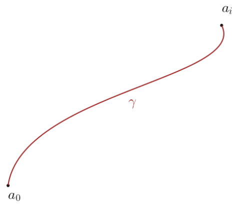

Figure: geodesic connecting points

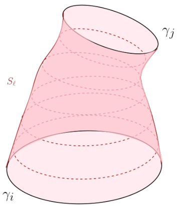

Figure: geodesic connecting closed curves

Approximation functional by penalisation

We define the **approximation functional** for Plateau's problem

$$F_\varepsilon(u) := \varepsilon \int_{C'} |\nabla u|^2 dx + \frac{1}{4\varepsilon} \int_{C'} (1-u)^2 dx + \frac{1}{c_\varepsilon} \sum_{i,j} d_u(\gamma_i, \gamma_j). \quad (1)$$

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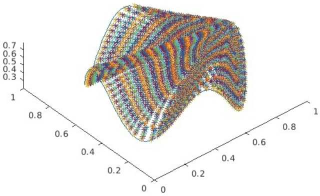

Figure: Path connecting Γ to γ_0

- $x_0 \in \Gamma$: fixed point in Γ.
- $\gamma_0(t) = x_0$: constant closed curve

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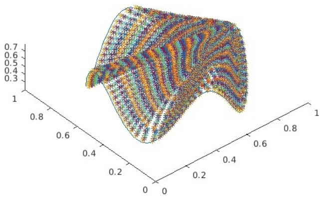


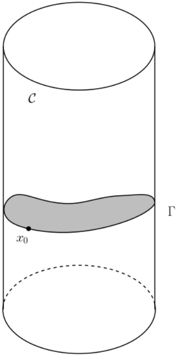
Figure: Path connecting Γ to γ_0

- $x_0 \in \Gamma$: fixed point in Γ.
- $\gamma_0(t) = x_0$: constant closed curve
- Penalization term for 1 closed curve : $\frac{1}{c_{\varepsilon}} d_u(\gamma_0, \Gamma)$

Limit Problem and Γ-convergence

Plateau in a cylinder

- C: the cylinder
- Γ: graph of a Lipschitz function, defined on the boundary of the cylinder ∂C



De Giorgi approach in codimension 1:
Surfaces are represented as boundary of sets and we minimize the perimeter of sets instead of the area of surfaces.

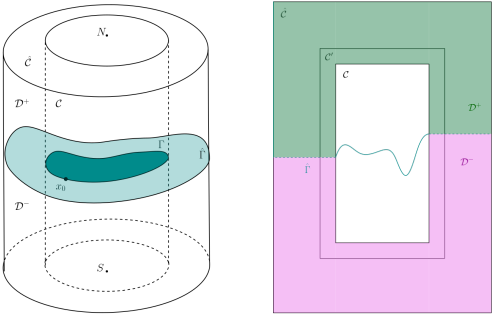


Figure: Interlocked cylinders

- $\hat{\Gamma}$: radial extension of the prescribed curve Γ
- $\mathcal{D}^+$: set above $\hat{\Gamma}$ in between cylinders
- $\mathcal{D}^-$: set below $\hat{\Gamma}$ in between cylinders

Surface spaning the curve Γ

↓

Set containing $\mathcal{D}^+$ and not meeting $\mathcal{D}^-$

Limit Problem

Definition (Competitor)

Ω is a competitor if it is a set of finite perimeter that satisfies the boundary constraint.

Definition (Plateau's problem)

$$\inf \{ P(\Omega, C') | \Omega \text{ a competitor.} \} \tag{2}$$

Remark

If Ω_0 is a solution of (2) $\partial^* \Omega_0$ is called the optimal surface.

Limit Problem

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Proposition (Existence)

Plateau's problem (2) *admits solutions.*

Regularity

Theorem (M., in preparation)

Let Ω_0 be a competitor, such that $\mathcal{H}^2$-a.e., $\partial^*\Omega_0 = \partial\Omega_0$. If Ω_0 is a minimizer of Plateau's problem then, it is a bi-John domain with Ahlfors regular boundary.

Regularity

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Remark: The regularity result remains valid in any dimension (in co-dimension 1). Furthermore if we consider quasi-minimizers (cf David and Semmes) instead of minimizer we have a characterization. Namely, the converse implication is also true.

Definition (Plateau-quasi-minimizer)

A competitor Ω_0 is called a Plateau-quasi-minimizer if there exists a constant $Q \geqslant 1$ such that, for all Ω competitor, we have

$$\mathcal{H}^2((\partial^*\Omega_0 \setminus \partial^*\Omega) \cap \hat{C}) \leqslant Q\mathcal{H}^2((\partial^*\Omega \setminus \partial^*\Omega_0) \cap \hat{C}).$$

Limsup and liminf result

Theorem (M., Bonnivard, Bretin, Lemenant, in preparation)

Let $\Omega \subset \hat{\mathcal{C}}$ *a competitor for Plateau's problem* (2). *Then, there exist* $(u_\varepsilon) \in H^1(\hat{\mathcal{C}}) \cap C(\overline{\hat{\mathcal{C}}})$ *such that* $0 \leqslant u_\varepsilon \leqslant 1$, $u_\varepsilon = 1$ *on* $\overline{\hat{\mathcal{C}}} \backslash \mathcal{C}'$ *and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leqslant \mathcal{H}^2(\partial^* \Omega \cap \overline{\mathcal{C}}).$$

Theorem (M., Bonnivard, Bretin, Lemenant, in preparation)

Let Ω *be a solution of Plateau's problem* (2). *Then, for all sequences* $u_\varepsilon \in H^1(\hat{\mathcal{C}}) \cap C^0(\overline{\hat{\mathcal{C}}})$ *such that* $0 \leqslant u_\varepsilon \leqslant 1$ *and* $u_\varepsilon = 1$ *on* $\overline{\hat{\mathcal{C}}} \backslash \mathcal{C}'$, *we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geqslant \mathcal{H}^2(\partial^* \Omega \cap \overline{\mathcal{C}}).$$

Γ-convergence type consequence

In the proof of the liminf Theorem, we constructed two competitors Ω^1 and Ω^2, as limit of level set of (u_ε), such that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geqslant \frac{1}{2}(\mathcal{H}^2(\partial^*\Omega^1 \cap \overline{C}) + \mathcal{H}^2(\partial^*\Omega^2 \cap \overline{C})).$$

Theorem (M., Bonnivard, Bretin, Lemenant, in preparation)

If we further assume the sequence (u_ε) to be quasi-minimal for F_ε then, the sets Ω^1 and Ω^2 are minimizers of Plateau's problem (2).

Definition (Quasi-minimizing sequence for F_ε)

We say that a sequence $u_\varepsilon \in H^1(\hat{C}) \cap C(\overline{\hat{C}})$ such that $0 \leqslant u_\varepsilon \leqslant 1$ and $u_\varepsilon = 1$ in $\overline{\hat{C}} \setminus C'$ is a quasi-minimal sequence for F_ε if

$$F_\varepsilon(u_\varepsilon) - \inf_u F_\varepsilon(u) \xrightarrow[\varepsilon\to 0]{} 0. \tag{3}$$

Numerical simulations

Numerical scheme for Steiner problem

We relax the geodesic distance between points :

$$G_\varepsilon(u,\gamma) = \varepsilon \int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (1-u)^2 dx + \frac{1}{c_\varepsilon}\sum_{i=1}^{N}\int_{\gamma_i} |u|^2 d\mathcal{H}^1,$$

where, $\gamma = (\gamma_i)_{1\leqslant i\leqslant N}$ and each γ_i is a Lipschitz curve connecting a_0 to a_i.

$$\begin{cases} \partial_t u = -\nabla_u G_\varepsilon(u,\gamma) \\ \gamma = Argmin_\gamma\{G_\varepsilon(u,\gamma)\}, \end{cases}$$

We use a time-decoupled scheme which alternates between

- a geodesic computation using the Fast Marching Method
- and a gradient descent to optimize in u.

☛ **Numerical approximation of the Steiner problem in dimension 2 and 3**, M. Bonnivard,E. Bretin and A. Lemenant, *Mathematics of Computation, 89, 1-43 (2020)*.

Numerical scheme for Plateau problem

We adapt the method implemented for Steiner to Plateau's problem.

$$G_\varepsilon(u,\ell) = \varepsilon \int_{C'} |\nabla u|^2 dx + \frac{1}{4\varepsilon} \int_{C'} (1-u)^2 dx + \frac{1}{c_\varepsilon} \int_{S_\ell} |u|^2 d\mathcal{H}^2.$$

We use a time-decoupled scheme which alternates between

- a relaxed geodesic computation using the Fast Marching Method
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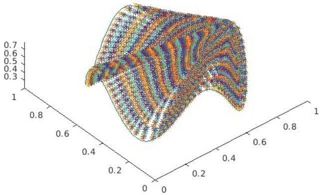
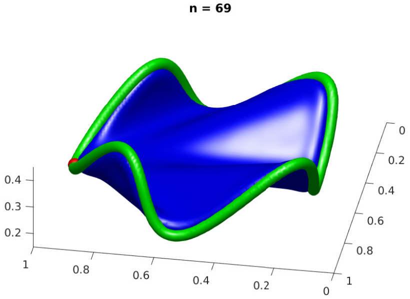
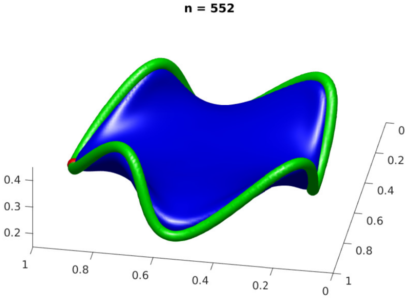


Figure: Relaxed geodesic connecting Γ to γ_0

Sinusoidal boundary





Connecting two circles

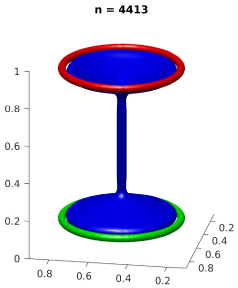


Figure: two distant circles

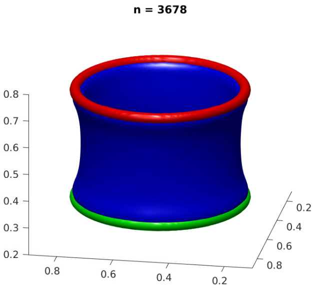
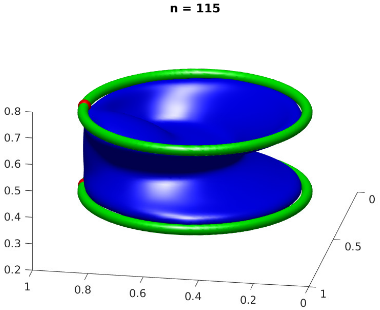
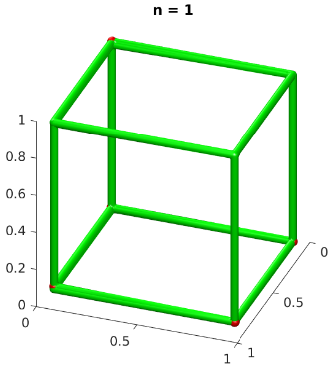


Figure: two close circles

Connecting a circle to a point on another circle



Cube



Thank you for your attention !

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Interfaces and free Boundaries, 20(1):69–106, 2018.

Eve Machefert.
Optimal regularity up to the boundary for plateau quasi-minimizers in a cylinder, *Preprint*.

Numerical scheme for the minimization of u

We want to solve $\nabla_u E^2_\varepsilon(u,\varphi) = 0$. To that aim we decompose

$$\nabla_u E^2_\varepsilon(u,\varphi) = J_{imp}(u,\varphi) + J_{exp}(u,\varphi),$$

in which we add αu to $J_{imp}(u,\varphi)$ and deduct from $J_{exp}(u,\varphi)$. So that for α big enough, J_{exp} is concave.
Then we get a semi-implicite scheme as follows

$$J_{imp}(u^{n+1},\varphi) + J_{exp}(u^n,\varphi) = 0.$$

we deal with the implicite term with the Fourier transform.