Periodic homogenization and harmonic measures

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Introduction: L^p-solvability of the Dirichlet problem

Consider the Dirichlet problem

$$\begin{cases} -\nabla \cdot A\nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

with A a uniformly elliptic and bounded matrix field and Ω the upper-half plane, that is, $\Omega = \mathbb{R}^{d+1}_+ := \{(t, x) \in \mathbb{R}^{d+1} : t > 0\}$.

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This requires $f \in H^{\frac{1}{2}}(\partial\Omega)$ since $\operatorname{Tr} H^1(\Omega) = H^{\frac{1}{2}}(\partial\Omega)$.

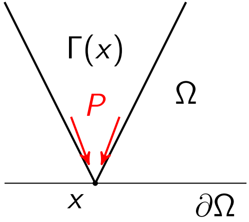
Introduction: L^p-solvability of the Dirichlet problem

Theorem ([Caf+81])

If u satisfies $Lu = -\nabla \cdot A\nabla u = 0$ in Ω, then the non-tangential limit

$$\lim_{\substack{P\in\Gamma(x),\\ P\to x}} u(P)$$

exists for almost every $x \in \partial\Omega$ wrt to the harmonic measure ω associated with the operator L and Ω.



Introduction: L^p-solvability of the Dirichlet problem

For a fixed $X \in \Omega$, since the linear application

$$f \in C_c(\partial\Omega) \mapsto u_f(X) \in \mathbb{R}$$

is bounded and positive by the maximum principle, the Riesz representation theorem provides the existence of a measure ω so that

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The L^p-solvability for some p is equivalent to $\omega \in A_\infty(\sigma)$,that is, the existence of $\varepsilon, \delta \in (0,1)$ such that for any $E \subset Q \subset \partial\Omega$

$$\frac{\sigma(E)}{\sigma(Q)} \leqslant \delta \quad \Rightarrow \quad \frac{\omega(E)}{\omega(Q)} \leqslant \varepsilon.$$

Dahlberg-Kenig-Pipher (DKP) condition

When $A = \mathrm{Id}$, that is, $L = -\Delta$, this property is verified.

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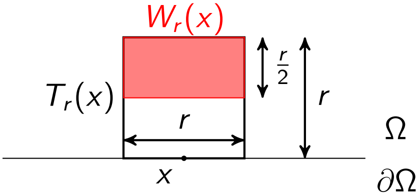
Define

$$T_r(x) := (0, r] \times Q_r(x), \quad W_r(x) := (r/2, r] \times Q_r(x).$$

Here $Q_r(x) \subset \partial\Omega = \mathbb{R}^d$ is the cube centered at x of side-length r.



Define the (local) oscillation of A at point $(t, y) \in \Omega$ by

$$\Big(\operatorname{osc} A\Big)(t, y) := \sup_{Y, Y' \in W_t(y)} |A(Y) - A(Y')|.$$

Dahlberg-Kenig-Pipher (DKP) condition

Theorem ([KP01])

If there is a constant $C > 0$ such that for all $r > 0$ and $x \in \partial\Omega$ one has

$$\int_{T_r(x)} |\operatorname{osc} A|^2(t,y)\frac{dydt}{t} \leqslant Cr^d,$$

then $\omega \in A_\infty(\sigma)$.

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This implies that $\operatorname{osc} A$ vanishes at a certain speed when $t \to 0$:

$$\int_{T_r(x)} |\operatorname{osc} A|^2(t,y)\frac{dydt}{t} = \int_0^r \left(\int_{Q_r(x)} |\operatorname{osc} A|^2(t,y)dy \right) \frac{dt}{t}.$$

Therefore, the DKP condition can be viewed as a reasonable perturbation of the Laplacian.

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Moreover, this result is sharp in the sense of convergence speed: If

$$\int_{T_r(x)} a^2(t,y)\frac{dydt}{t} = +\infty,$$

then there exists a uniformly elliptic matrix field A such that $\operatorname{osc} A \sim a$ and the harmonic measure associated to the operator $-\nabla \cdot A\nabla$ does not admit the A_∞ property.

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Question: Can homogenization be used to replace the assumption of vanishing oscillations in the DKP condition?
Answer: Yes!

Main theorem

Suppose that $A_\#$ is a $\mathbb{Z}^{d+1}$ periodic elliptic matrix field on $\mathbb{R}^{d+1}$ with associated homogenized matrix $\bar{A}_\# = \mathrm{Id}$. We define the coefficient field A in the following way

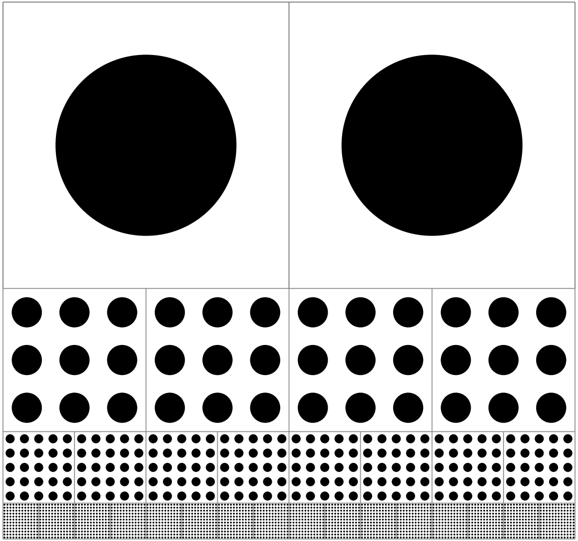
$$A(t,x) = \begin{cases} \mathrm{Id} & t \geqslant 1, \\ A_\#(\frac{t,x}{2^k \varepsilon_k}) & t \in [2^{k-1}, 2^k), k \leqslant 0. \end{cases}$$

Theorem ([Dav+25])

When $\varepsilon_k \lesssim 2^k$, $\omega \in A_\infty(d\sigma)$.

In particular, if $A_\#$ is not constant, the DKP condition cannot be satisfied.

Main theorem



Proof of the main theorem

A criterion for the A_∞ property of the harmonic measure is the following:

Theorem ([Ken+00])

The harmonic measure $\omega \in A_\infty(d\sigma)$ if and only if for any continuous solution u to the Dirichlet problem,

$$r^{-d} \int_{T_r(x)} t|\nabla u(t,y)|^2 dydt \leqslant C\|f\|_{L^\infty}^2.$$

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Ansatz:

$$u \sim u^{2s} := \bar{u} + \sum_{k \leqslant 0} 2^k \varepsilon_k \chi_k \phi_i\left(\frac{\cdot}{2^k \varepsilon_k}\right)\partial_i \bar{u},$$

where χ_k is a cut-off because of the boundary layer, $\bar{u}$ is the solution to the Dirichlet problem with the same boundary datum f and $A = \bar{A}_\# = \mathrm{Id}$.

Proof of the main theorem

The key estimate is that

$$r^{-d}\int_{T_r(x)} t|\nabla z(t,y)|^2 dydt \leqslant C\|f\|_{L^\infty}^2,$$

where $z := u - u^{2s}$ is the homogenization error. It satisfies the following equation:

$$-\nabla\cdot A\nabla z = \nabla\cdot f,$$

with

$$f := \sum_{k\leqslant 0} 2^k\varepsilon_k(A_\#\phi^i - \sigma^i)(\tfrac{\cdot}{2^k\varepsilon_k})\nabla(\chi_k\partial_i\bar{u}) + \mathbf{1}_{t\in[2^{k+1},2^k]}(1-\chi_k)(A_\# - \mathrm{Id})(\tfrac{\cdot}{2^k\varepsilon_k})\nabla\bar{u}.$$

Proof of the main theorem

We localize the error in $T_r(x)$ by Caccioppoli's inequality, maximum principle and carefully choosing the boundary layer size:

$$\begin{aligned}\int_{T_r(x)} |\nabla z|^2 &\lesssim \int_{T_{2r}(x)} f^2 + r^{-2} \int_{T_{2r}(x)} z^2 \\ &\lesssim \sum_{k \leqslant \min([\log_2 r],0)} \varepsilon_k \int_{[2^{k+1},2^k)\times Q_r(x)} |\nabla \bar{u}|^2 + r^{d-1}.\end{aligned}$$

Therefore, if $\varepsilon_k \lesssim 2^k$, then one can deduce, by harmonicity of $\bar{u}$,

$$\int_{T_r(x)} \min(t,1)|\nabla z|^2 \lesssim \min(r,1)(\underbrace{\int_{T_r(x)} t|\nabla \bar{u}|^2}_{\lesssim r^d} + r^{d-1}) \sim r^d.$$

Further questions

It's interesting to understand how the speed $\varepsilon_k \lesssim 2^k$ in the assumption can be interpreted or improved.

1. If we pick the coefficient field A differently in each layer $[2^{k+1}, 2^k)$ (instead of a single $A_{\#}$), then we need a stronger assumption $\varepsilon_k \lesssim 2^{\frac{3}{2}k}$.
2. If there exists a global solution for the corrector equation, then the weaker assumption $\varepsilon_k \lesssim 2^{\frac{1}{2}k}$ is enough.
3. Can we go further and prove the theorem for $\sum_{k\leqslant 0} \varepsilon_k^q < +\infty$?

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