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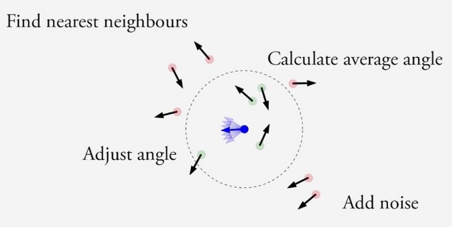
Numerical approximation of phase transition in collective dynamics

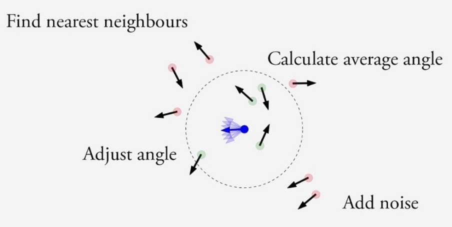
Joint work with Pierre Degond and Sara Merino

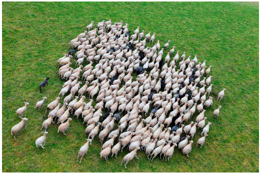
Léo Meyer

The Vicsek model



The Vicsek model





The Vicsek model

We consider a family of N particles described by their **positions** and **directions** $(x_i(t), \omega_i(t))_{i=1,\dots,N} \subset \mathbb{R}^d \times S^{d-1}$ and solving:

$$\begin{cases} \mathrm{d}x_i(t) = \omega_i(t)\mathrm{d}t, \\ \mathrm{d}\omega_i(t) = P_{\omega_i^\perp}\Big(\nu(|J_i|)\Omega_i \mathrm{d}t + \sqrt{2\mu(|J_i|)}\mathrm{d}B_i(t)\Big), \end{cases}$$

where $J_i = \frac{1}{N}\sum_{j=1}^N K(x_j, \omega_j, x_i, \omega_i)\omega_j$ and $\Omega_i = \frac{J_i}{|J_i|}$.

Mean-field limit

We proceed with the mean-field limit $N \to \infty$ to obtain the density probability of particles $f(t,x,\omega) : \mathbb{R}_+ \times \mathbb{R}^d \times S^{d-1} \to [0,1]$. It solves the following equation:

$$\partial_t f + \omega \cdot \nabla_x f + \nu(|J_f|)\nabla_\omega \cdot (P_{\omega^\perp}\Omega_f f) = \mu(|J_f|)\Delta_\omega f,$$

where $\hat{\omega}_f = \int_{S^{d-1}} \omega f \mathrm{d}\omega$, $J_f = K * \hat{\omega}_f$ and $\Omega_f = \frac{J_f}{|J_f|}$.

Hydrodynamic limit

Finally, using the change of variable $x = \varepsilon x$, $t = \varepsilon t$, we get the rescaled equation:

$$\varepsilon \left(\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon \right) = Q(f^\varepsilon),$$

where $Q(f) = -\nu(|J_f|)\nabla_\omega \cdot (P_{\omega^\perp}\Omega_f f) + \mu(|J_f|)\Delta_\omega f$. In the limit $\varepsilon \to 0$, we write $f^\varepsilon \to \bar{f}$, and $\bar{f} \in \mathrm{Ker}(Q)$.

The kernel of Q is described by the **von Mises distributions** $M_{\kappa\Omega}(\omega) = Z^{-1}\exp(\omega \cdot \kappa\Omega)$. In particular, we get $\bar{f}$ of the form $\bar{f} = \rho M_{\kappa\Omega}$, where $\rho, \kappa, \Omega : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and κ solves the consistency equation:

$$j(\kappa) = \rho c(\kappa),$$

where j is the inverse of $\frac{\nu}{\mu}$.

Phase transition

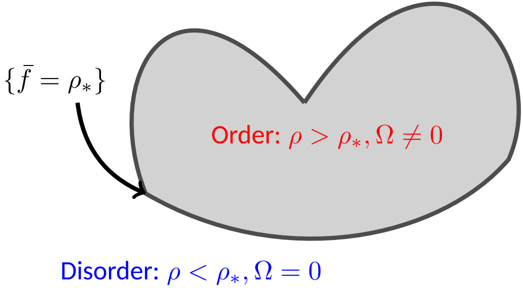
Theorem (Phase transition, Degond, Frouvelle, Liu (2015))

Let $\rho > 0$. Define $\rho_c = \lim_{\kappa \to 0} \frac{j(\kappa)}{c(\kappa)}$ and $\rho_* = \inf_{\kappa \in (0, \kappa_{\max})} \frac{j(\kappa)}{c(\kappa)}$ where $\rho_c > 0$ may be equal to $+\infty$ and $\kappa_{\max} = \lim_{|J| \to +\infty} \frac{\nu(|J|)}{\mu(|J|)}$.

1. *If $\rho < \rho_*$, the only solution to the compatibility equation is $\kappa = 0$ and the only equilibrium with total mass ρ is the **uniform distribution** $\bar{f} = \rho$.*
2. *If $\rho > \rho_*$, there exists **at least one positive solution** $\kappa > 0$ to the compatibility equation. It corresponds to a family $\{\rho M_{\kappa\Omega} | \Omega \in S^{d-1}\}$ of non-isotropic equilibria.*
3. *The number of families of non-isotropic equilibria changes as ρ crosses the threshold ρ_c.*

Illustration of phase transition

We only consider cases where $\rho_c = \rho_*$.



Two regimes of phase transition

Ordered region: Self-organised hydrodynamic model (SOH)

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho c_1(\rho)\Omega) = 0, \\ \rho(\partial_t \Omega + c_2(\rho)(\Omega \cdot \nabla_x)\Omega) + \Theta(\rho) P_{\Omega^\perp} \nabla_x \rho = \mathcal{K}_2 \delta(\rho) P_{\Omega^\perp} \Delta_x (\rho c_1 \Omega). \end{cases}$$

Disordered region:

$$\begin{cases} \partial_t \rho = \varepsilon \nabla_x \cdot \left(\frac{1}{1 - \frac{\rho}{\rho_*}} \nabla_x \rho \right), \\ \Omega = 0. \end{cases}$$

Goal: Simulate phase transition without tracking the boundary.

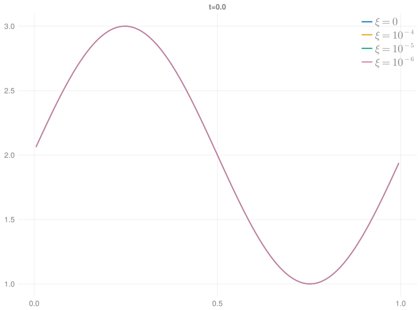
Meta-model

$$
\begin{cases}
\partial_t \rho + \nabla_x \cdot (\rho v) = 0, \\
\partial_t(\rho v) + \nabla_x p(\rho) + \nabla_x \cdot \left(\rho \left[\frac{c_2(\rho)}{c_1(\rho)} v + \mathcal{K}_2 \nabla_x((\delta c_1)(\rho))\right] \otimes v\right) \\
\qquad\qquad = \mathcal{K}_2 \nabla_x \cdot [(\delta c_1)(\rho) \nabla_x(\rho v)] - \frac{1}{\xi} \rho v \, h(\rho) \left(1 - \frac{c_1^2(\rho)}{|v|^2}\right),
\end{cases}
$$

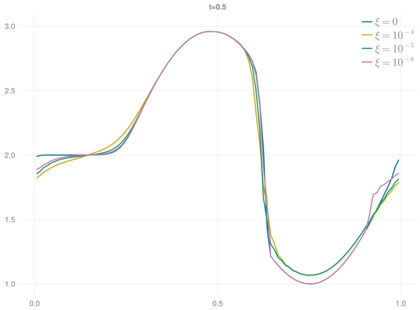
where $p' = c_1 \Theta$ and $h(\rho) = c|1 - \frac{\rho}{\rho_*}|$.
We make the following choice for ν and μ:

$$
\nu(|J|) = \frac{\nu_0 |J|}{|J| + J_0} \quad \text{and} \quad \mu(|J|) = \mu_0.
$$

Critical exemple



Critical exemple



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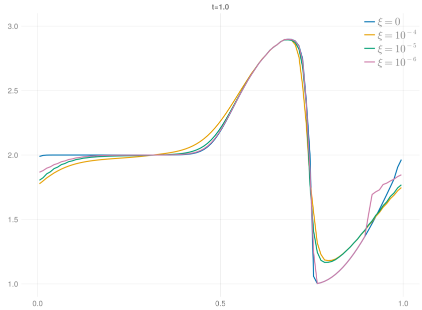
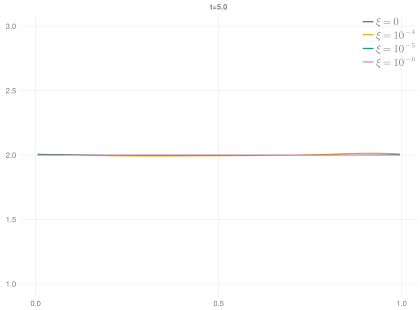
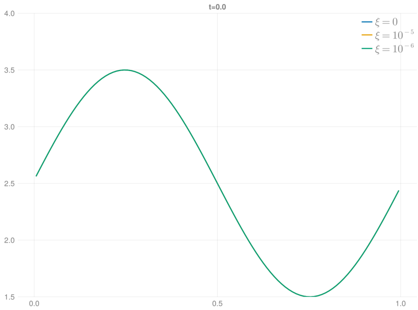


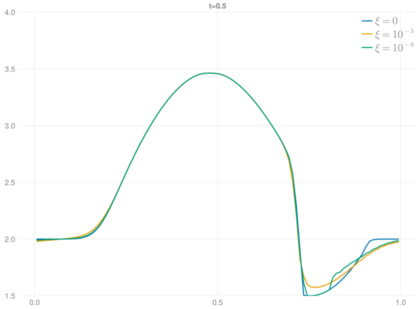
Critical exemple



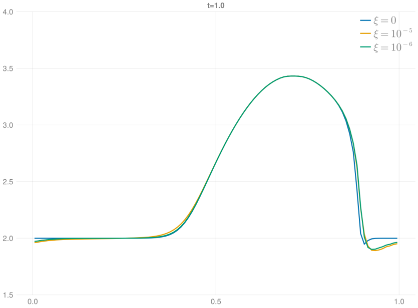
Super-critical exemple



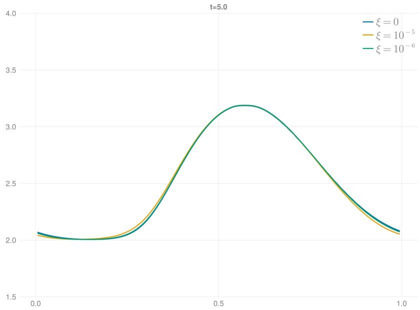
Super-critical exemple



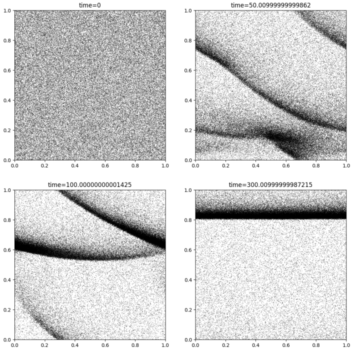
Super-critical exemple



Super-critical exemple



2D exemple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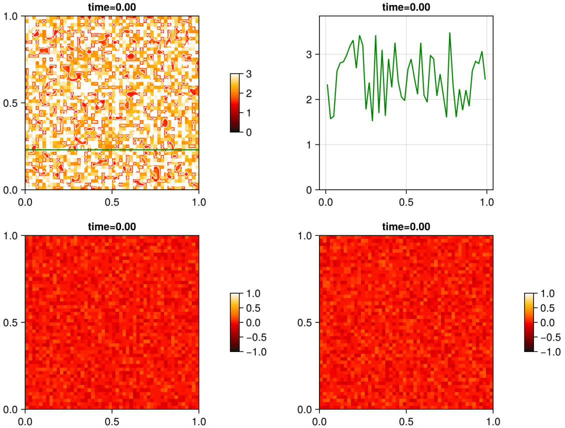
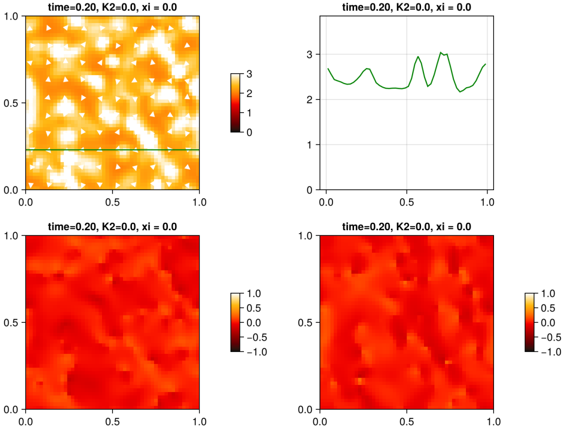
time=100.00000000001425
time=300.00999999987215

2D exemple

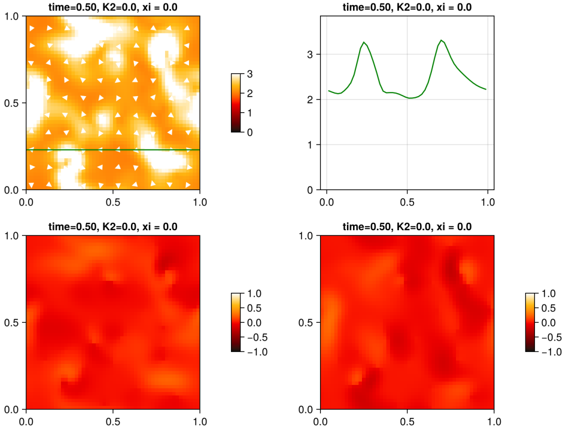


2D exemple



time=0.20, K2=0.0, xi = 0.0
time=0.20, K2=0.0, xi = 0.0
time=0.20, K2=0.0, xi = 0.0
time=0.20, K2=0.0, xi = 0.0

2D exemple



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time=0.50, K2=0.0, xi = 0.0
time=0.50, K2=0.0, xi = 0.0
time=0.50, K2=0.0, xi = 0.0
0.0
0.5
1.0
0
1
2
3
−1.0
−0.5

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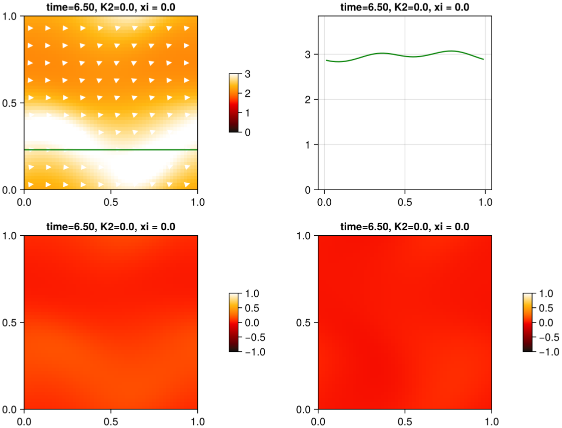
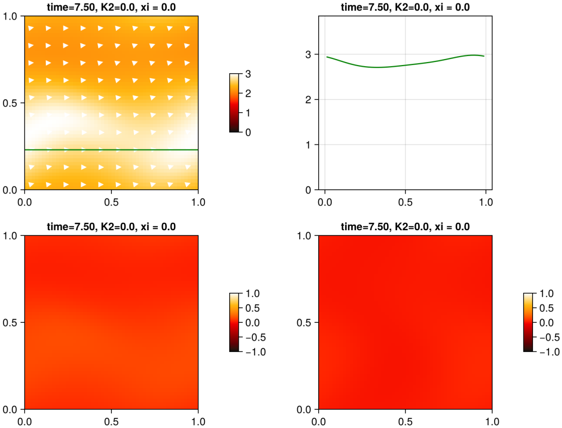
2D exemple



2D exemple



time=7.50, K2=0.0, xi = 0.0
time=7.50, K2=0.0, xi = 0.0
time=7.50, K2=0.0, xi = 0.0
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Perspectives

- Other complex behaviour at macro scale : mills, etc…
- Comparison with particle model
- Applying the method to other particle models
- Second order in space

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Thank you for your attention !