Modélisation de micro-filaments et convergence d’un modèle à N segments

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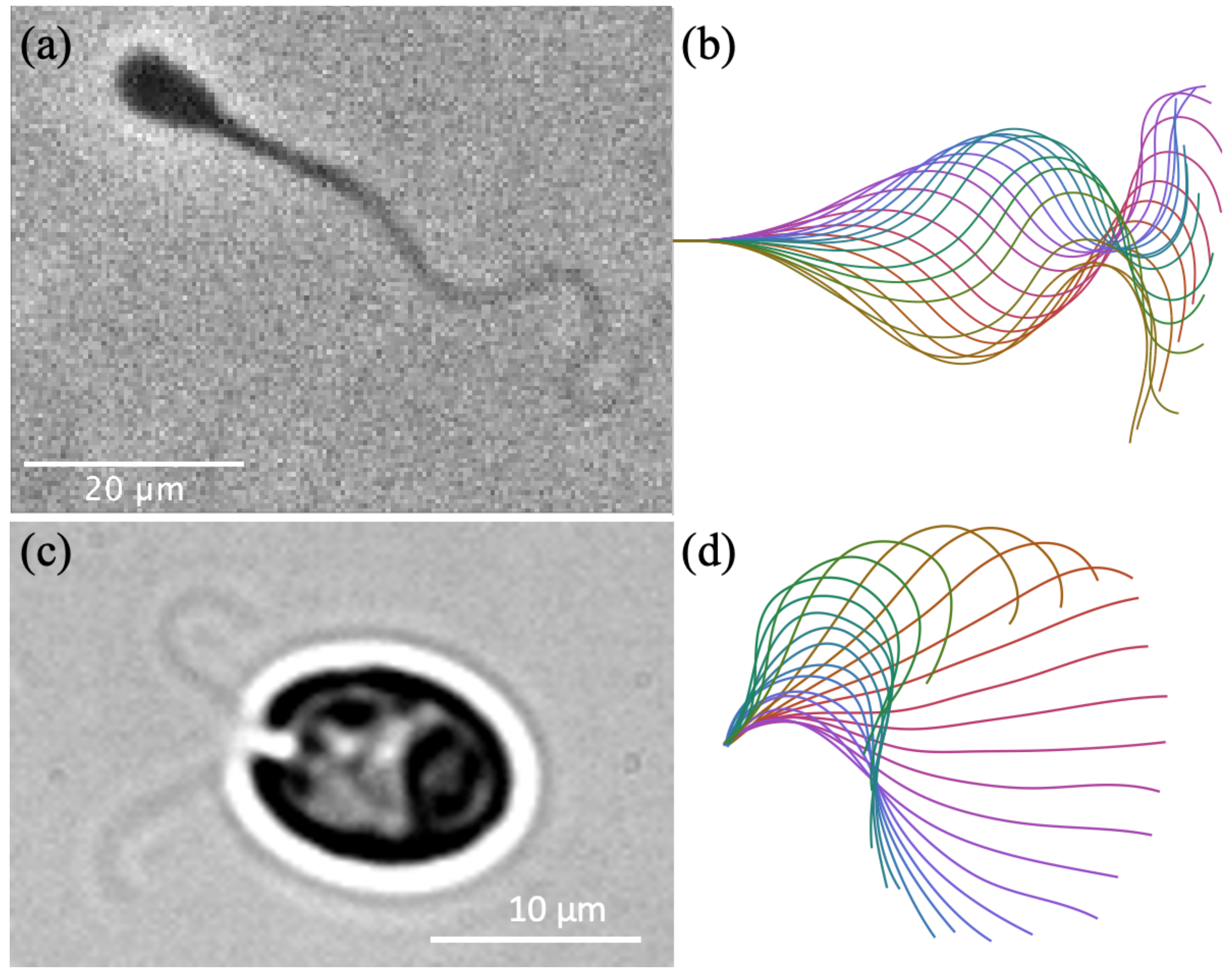
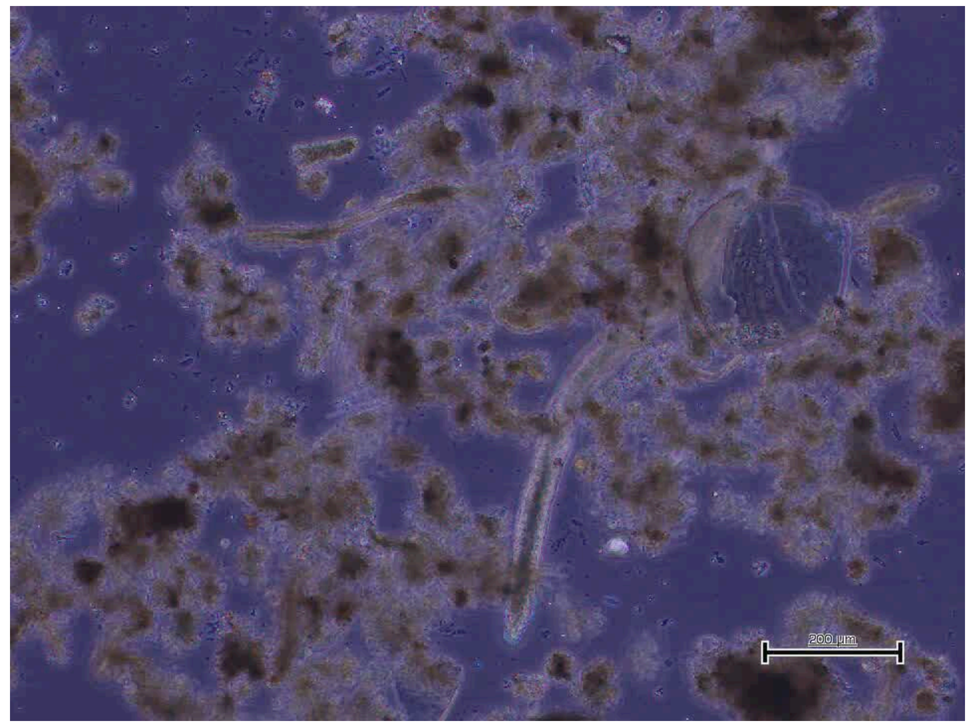
Congrès SMAI — 6 juin 2025

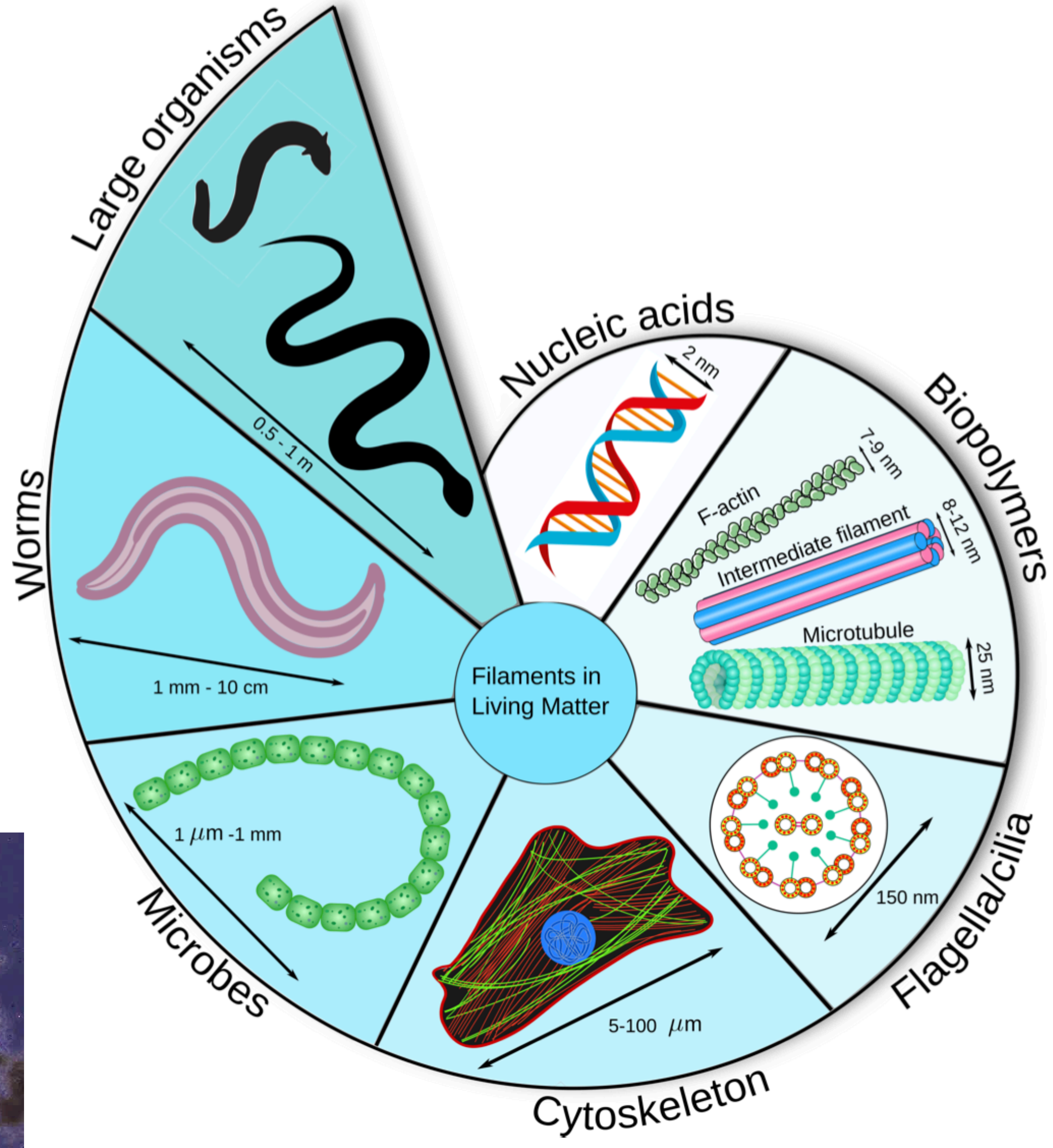
Flexible filaments are ubiquitous at microscopic scale

Moreau et al., Phys. Rev. Research 6 (2024)

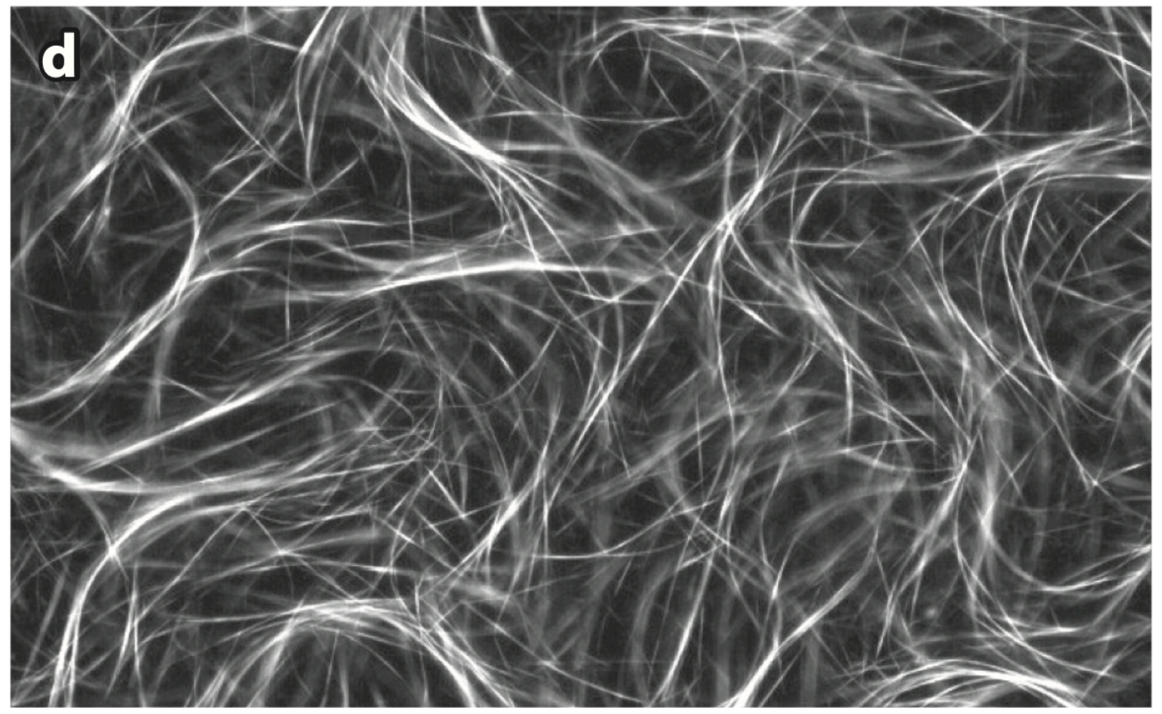


Kyoto University pond

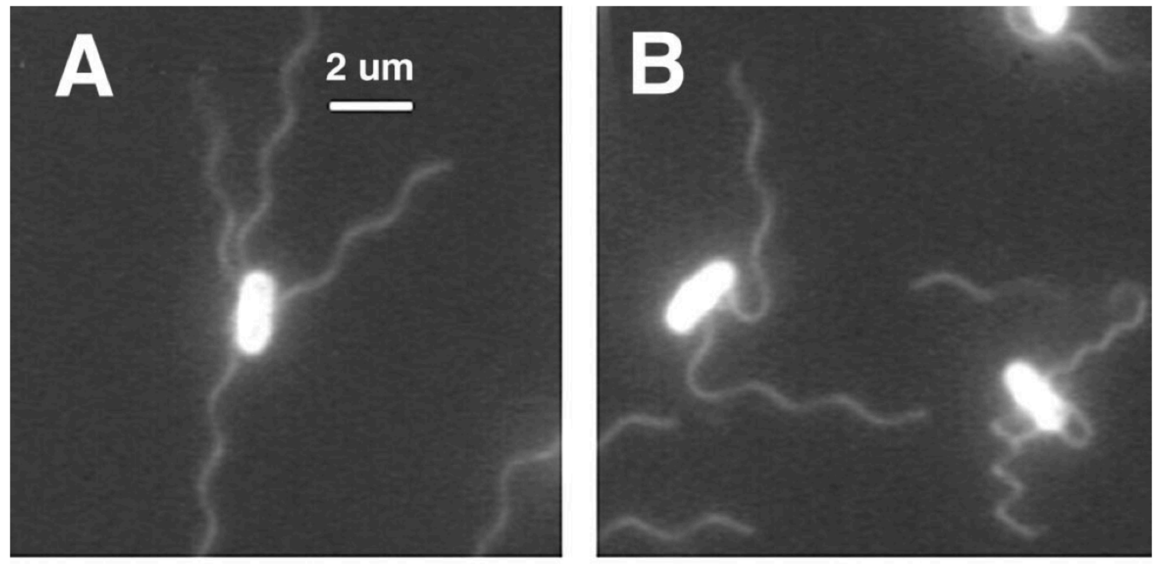




Cammann et al. arXiv:2502.12731



Sanchez et al., Nature 491 (2012)

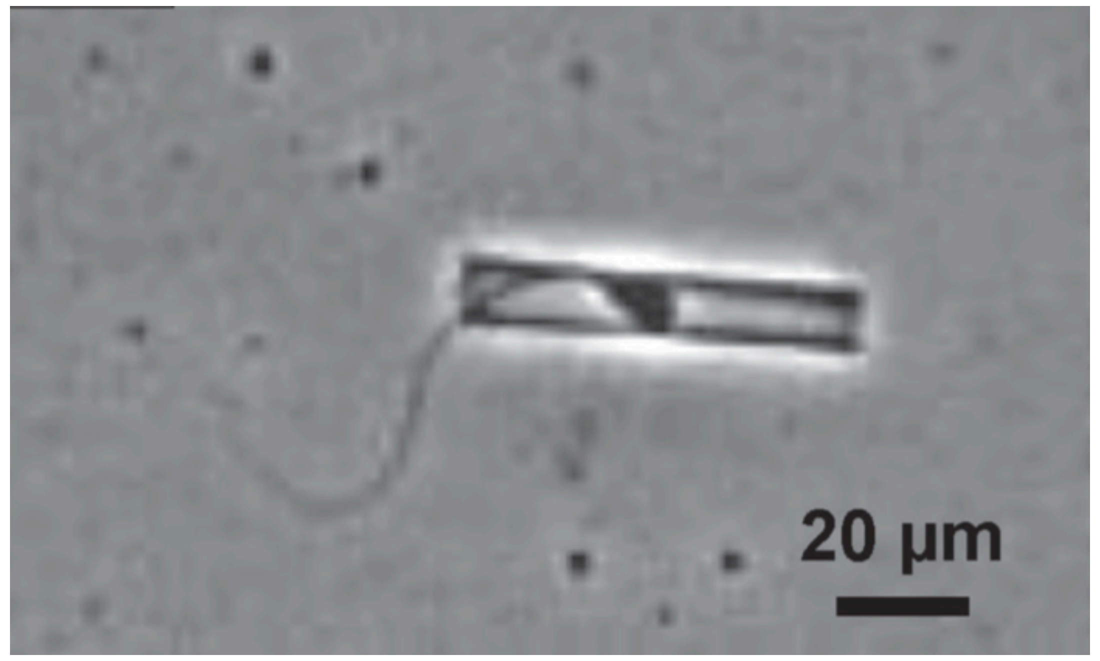


Turner et al., J. Bacteriol. 182 (2000)

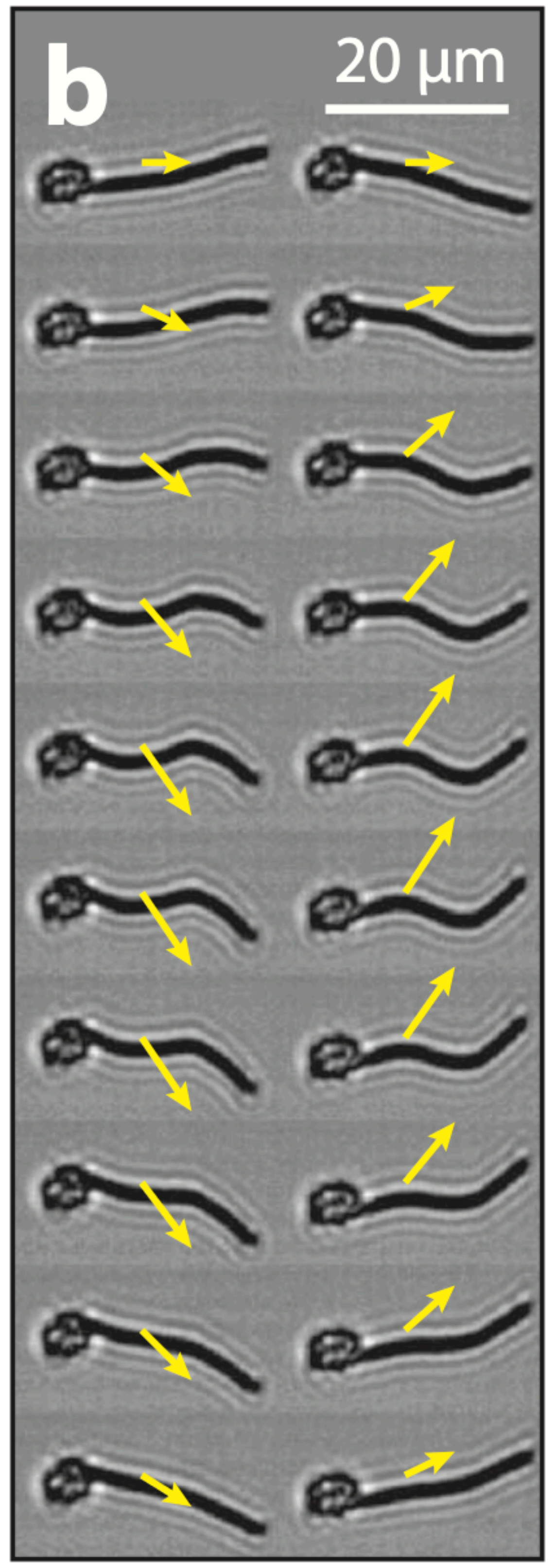
Dynamics : **elastic body** interacting with **fluid flow** (+ some kind of **activity**)

Bio-inspired slender micro-robots

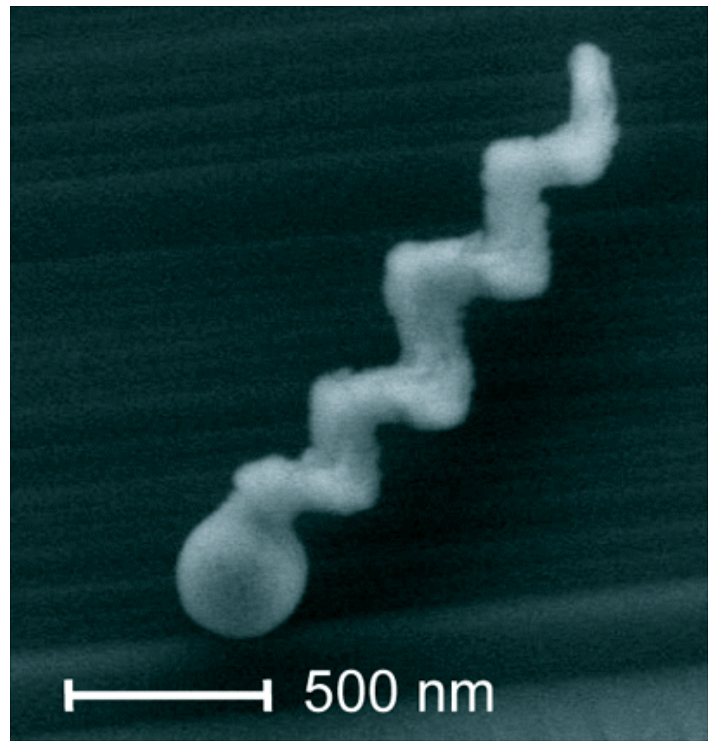
Magdanz et al., Adv. Funct. Mater. 25(18), 2015



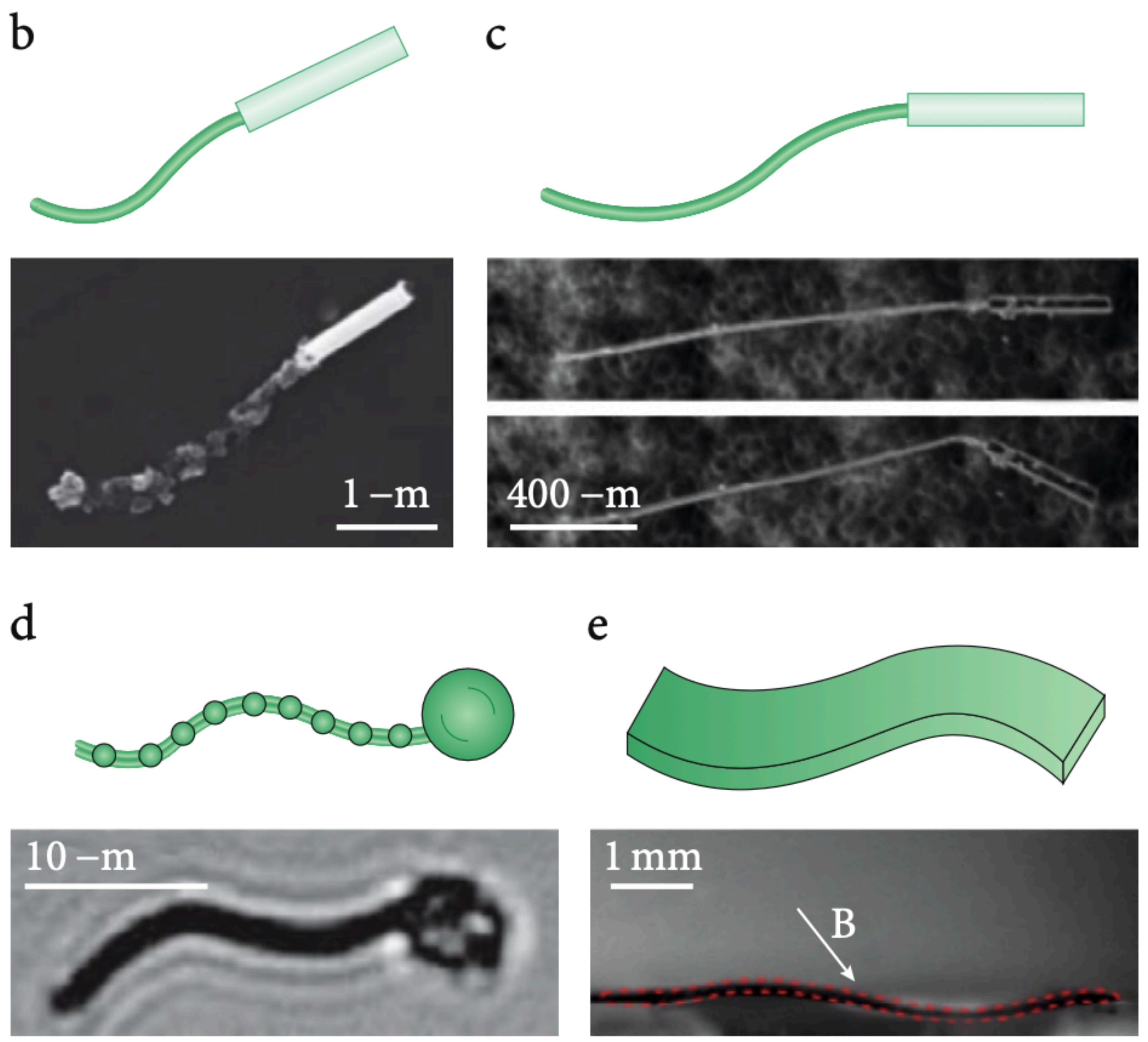
Palagi, Fischer, Nat. Rev. Materials 3 (2018)

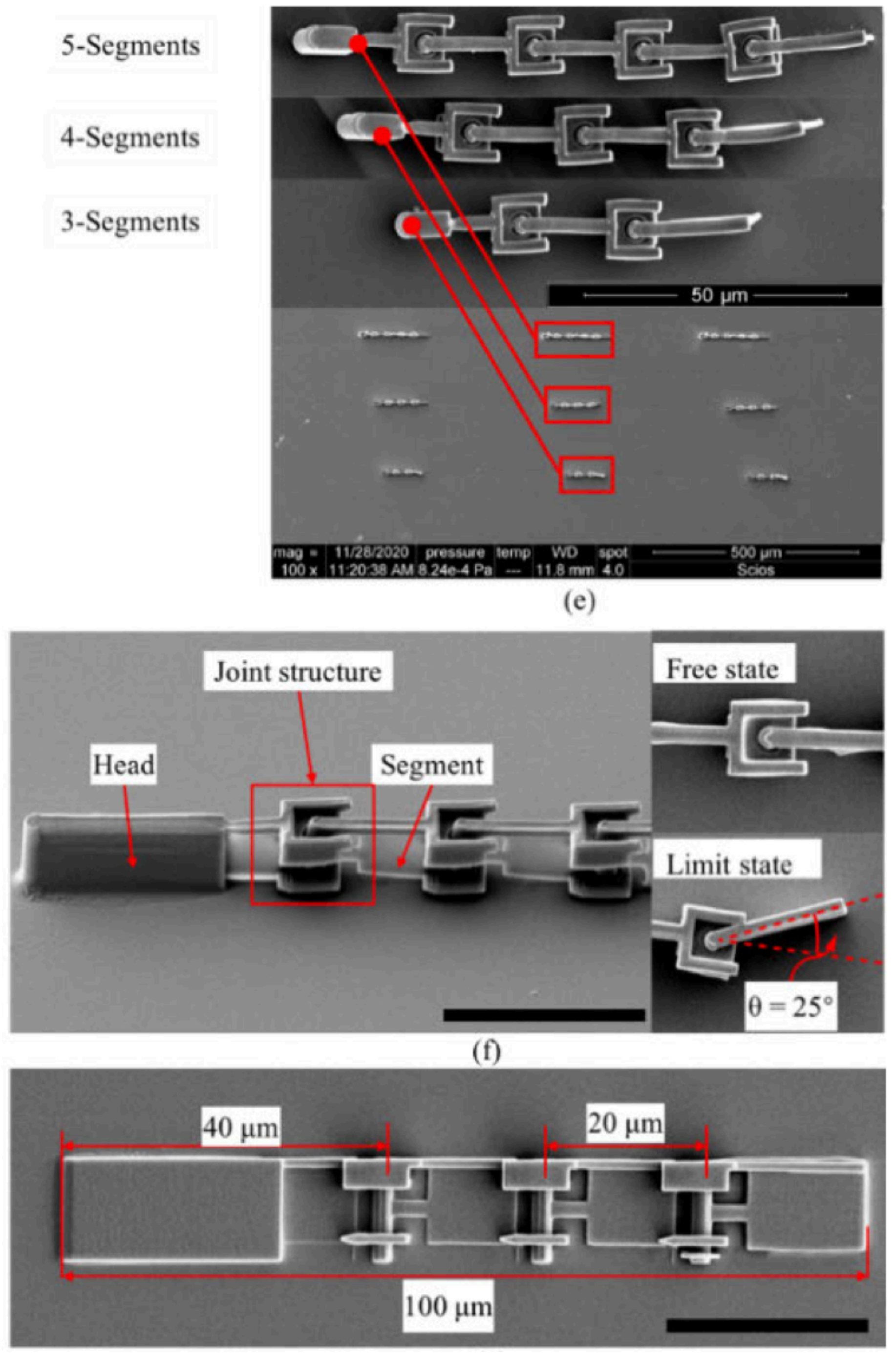


Xin et al., ACS Nano 15(11), 2021



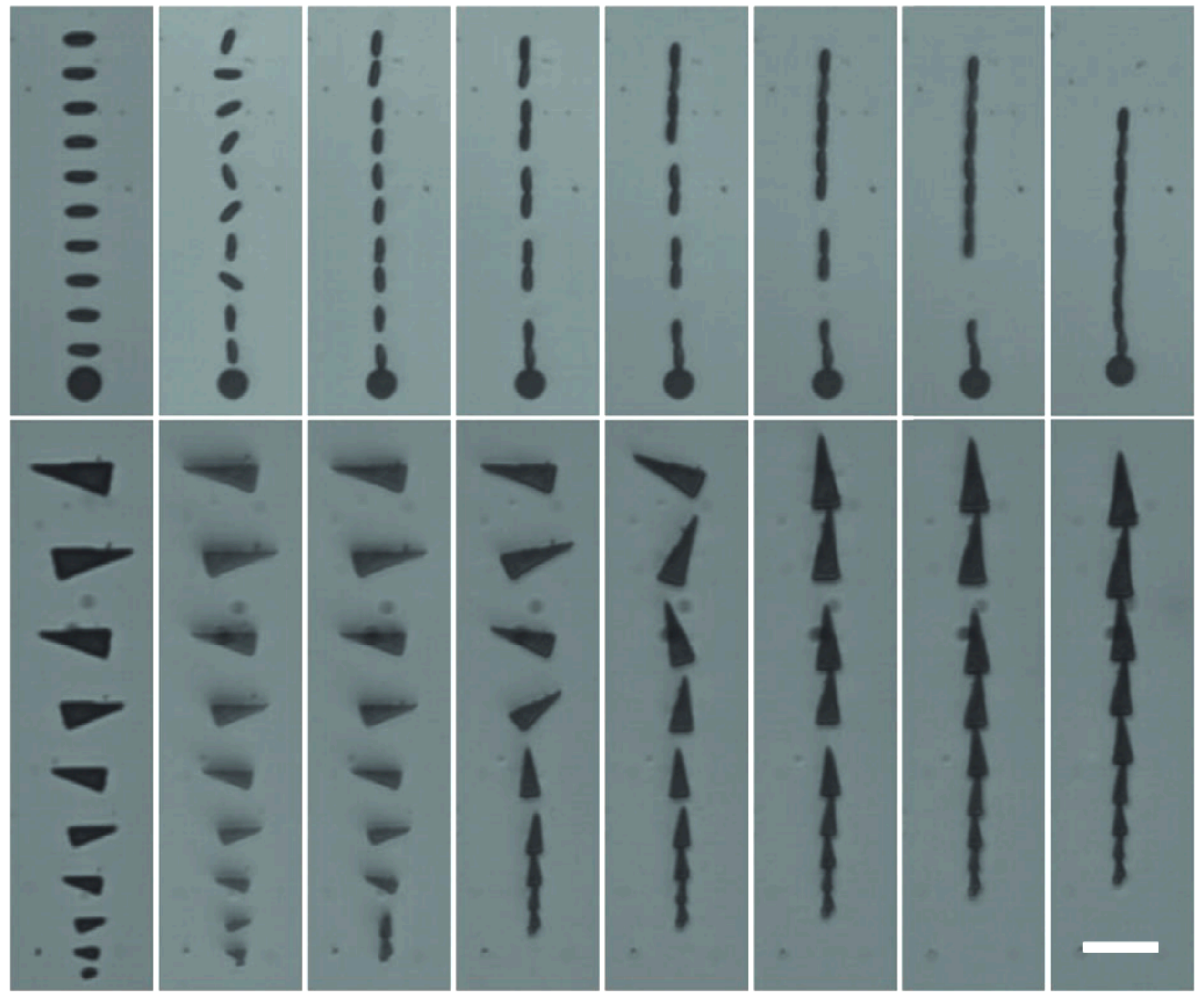
Ghosh & Fischer, Nano Letters 9(6), 2009





Xing et al. *IEEE Access* 9, 2021

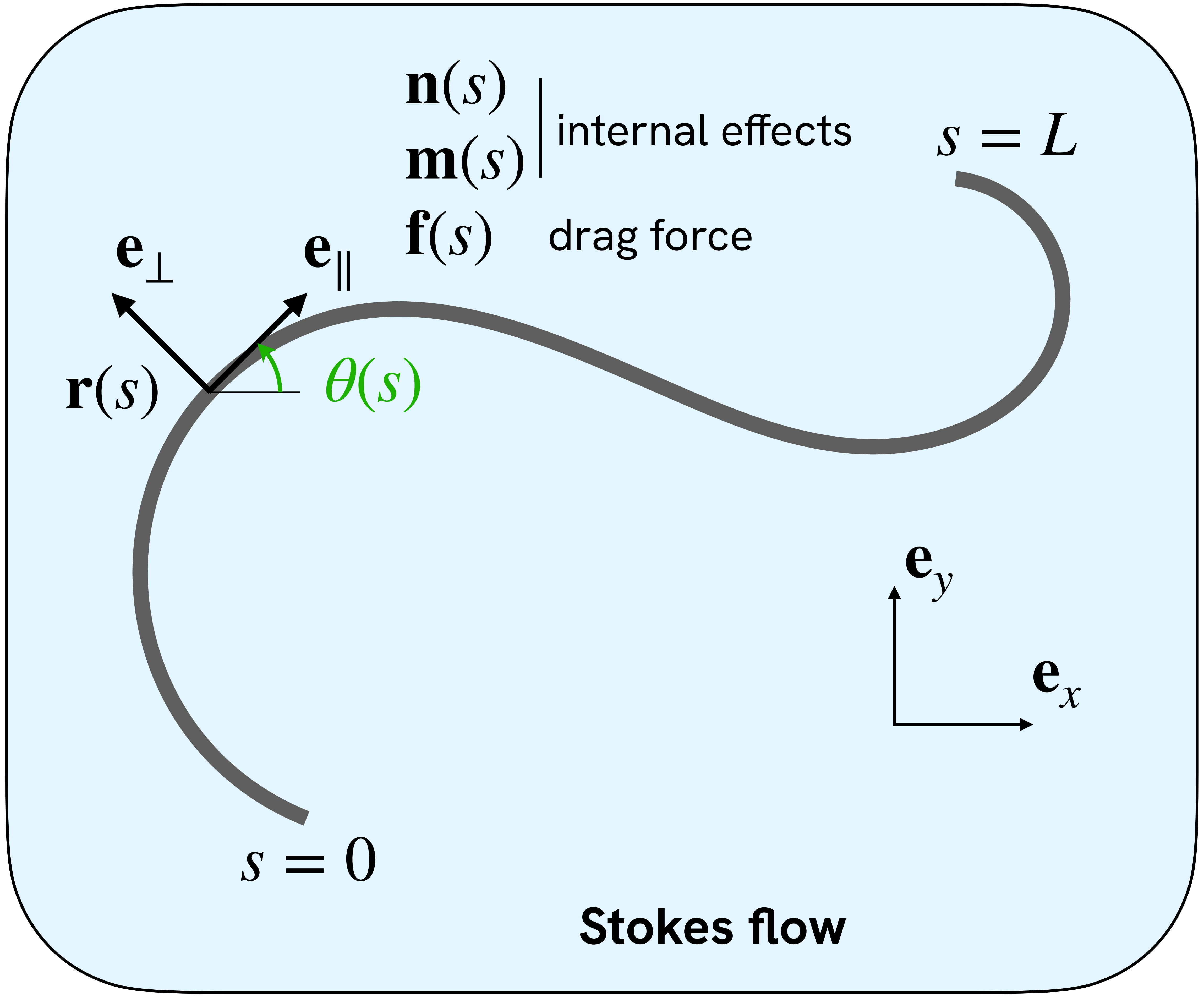
Pauer et al., Adv. Mater. 33, 2021



→ **modelling and simulation**

Elastohydrodynamics equations

Deformable slender body in a viscous **fluid** (Stokes flow)



Resistive Force Theory (RFT): local anisotropic operator

$$\mathbf{f}(t,s) = \mathbf{C}(\theta(t,s))\dot{\mathbf{r}} = -c_\parallel(\mathbf{e}_\parallel \cdot \dot{\mathbf{r}})\mathbf{e}_\parallel - c_\perp(\mathbf{e}_\perp \cdot \dot{\mathbf{r}})\mathbf{e}_\perp$$

$\mathbf{f}$: hydrodynamic force density; $c_\parallel(\mathbf{e}_\parallel \cdot \dot{\mathbf{r}})\mathbf{e}_\parallel$: tangential velocity; $c_\perp(\mathbf{e}_\perp \cdot \dot{\mathbf{r}})\mathbf{e}_\perp$: orthogonal velocity

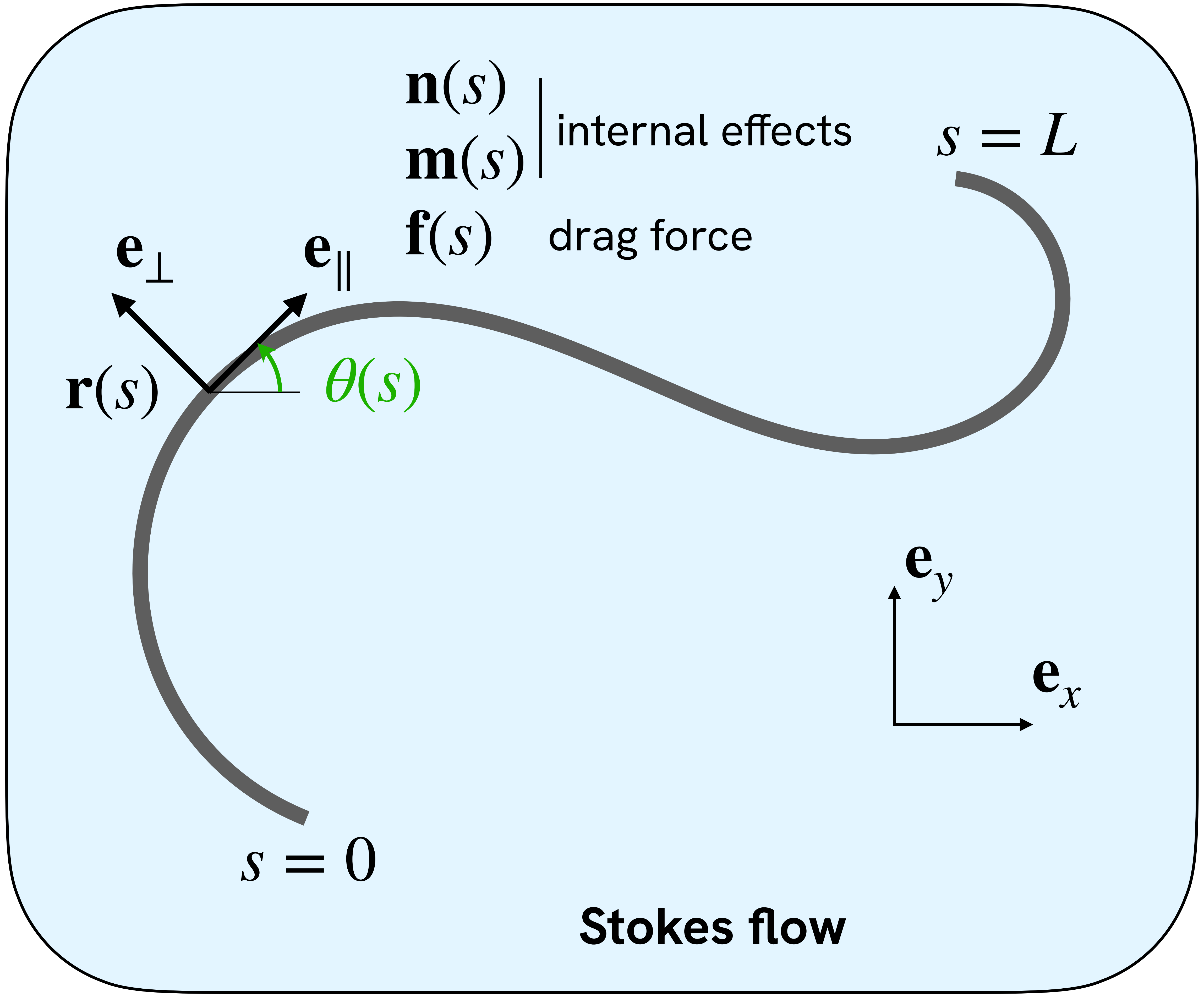
(Cont)

$\partial_s \mathbf{n} + \mathbf{f} = 0$	(local force balance)
$\partial_s \mathbf{m} + \mathbf{r}_s \times \mathbf{n} = 0$	(local torque balance)
$\mathbf{m} = E\theta_s \mathbf{e}_z$	(elastic constitutive law)
$\lvert \mathbf{r}_s \rvert^2 = 1$	(inextensibility)
+ B.C.	

📖 Antman, *Nonlinear Problems of Elasticity*, 2005

Elastohydrodynamics equations

Deformable slender body in a viscous **fluid** (Stokes flow)



(Cont)

$$\partial_s \mathbf{n} + \mathbf{C}(\theta)\partial_t \mathbf{r} = 0 \quad \text{(local force balance)}$$

$$\partial_s \mathbf{m} + \mathbf{r}_s \times \mathbf{n} = 0 \quad \text{(local torque balance)}$$

$$\mathbf{m} = E\theta_s \mathbf{e}_z \quad \text{(elastic constitutive law)}$$

$$|\mathbf{r}_s|^2 = 1 \quad \text{(inextensibility)}$$

+ B.C.

free	pinned	clamped
$\mathbf{m} = 0,\ \mathbf{n} = 0$	$\mathbf{m} = 0,\ \dot{\mathbf{r}} = 0$	$\dot{\theta} = 0,\ \dot{\mathbf{r}} = 0$

📖 Antman, *Nonlinear Problems of Elasticity*, 2005

Elastohydrodynamics equations

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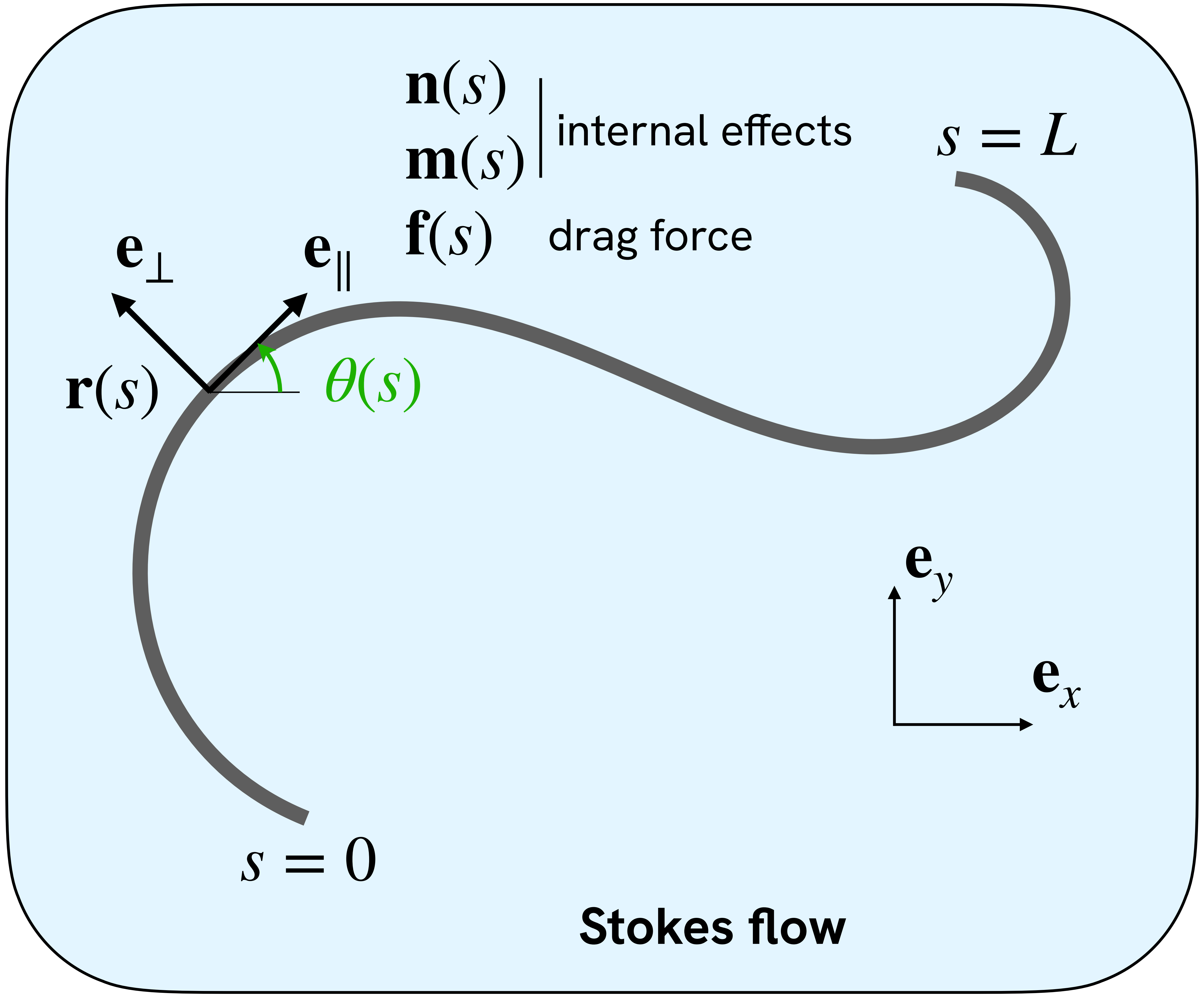

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+ B.C.

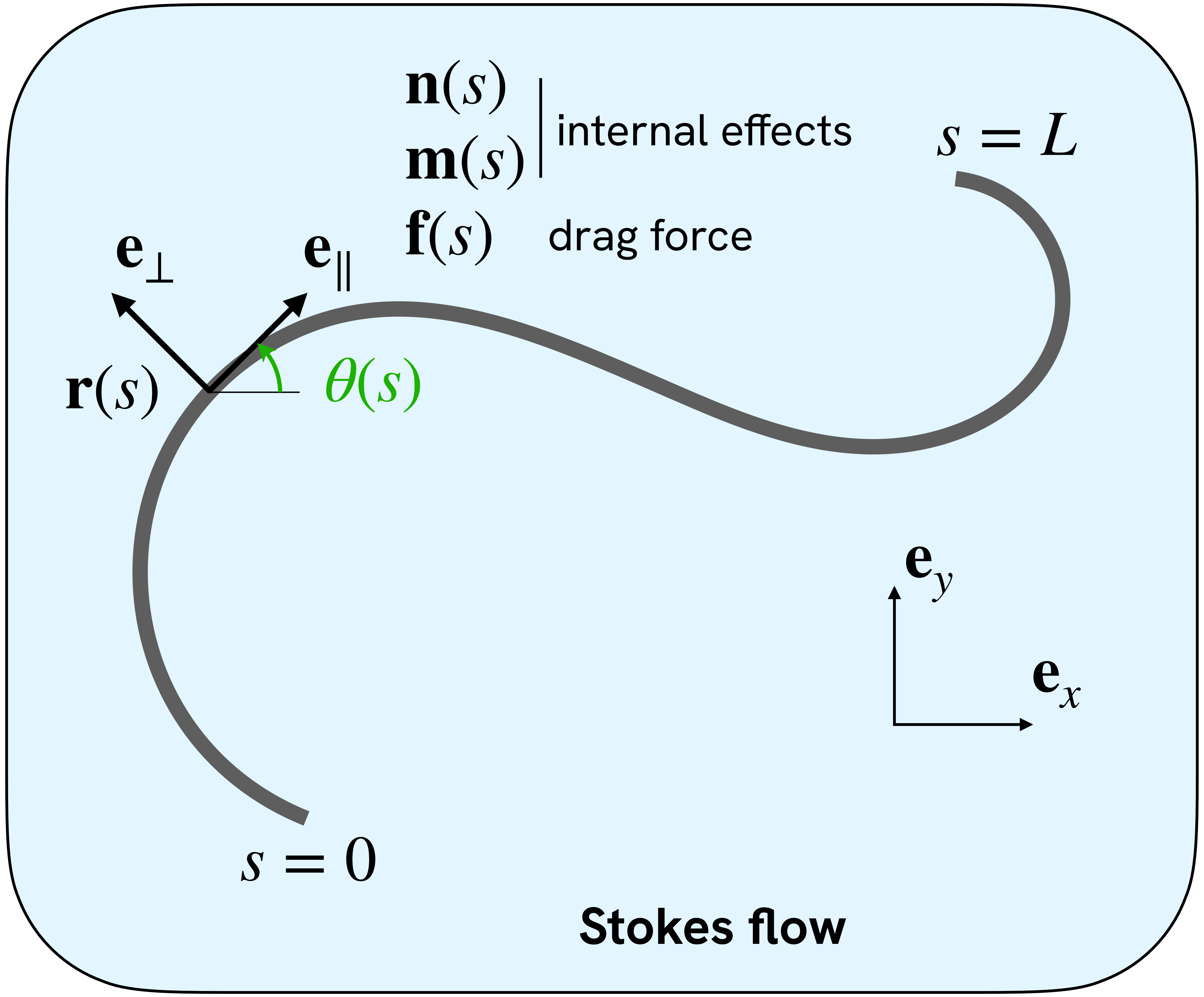
- **Nonlinear PDE** with 4th space derivative + τ (Lagrange multiplier for the tension)

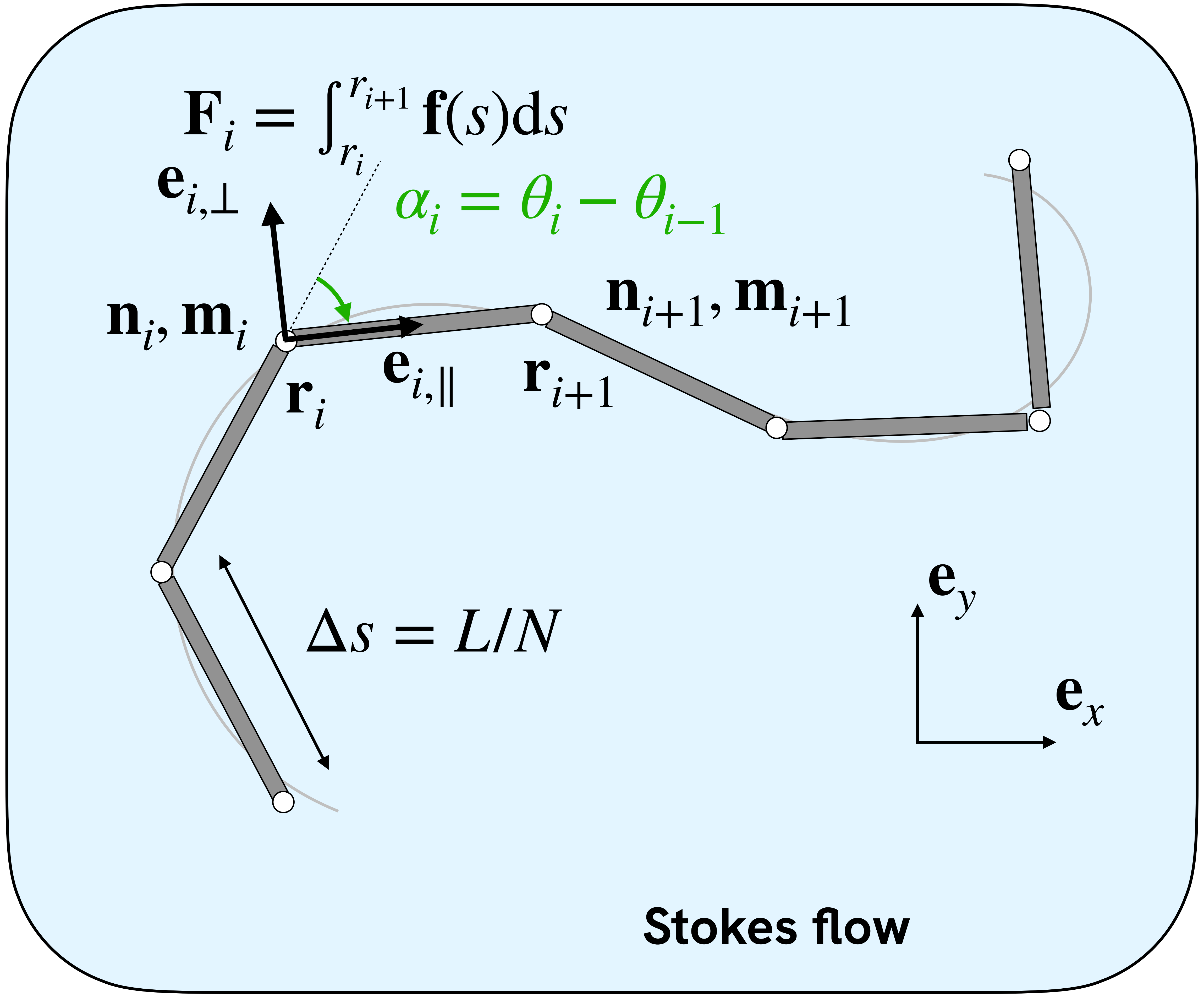
$$\dot{\mathbf{r}} = -(\mathbf{I} + \gamma \mathbf{r}_s \mathbf{r}_s^T)^{-1} \cdot \partial_s(\mathbf{r}_{sss} - \tau \mathbf{r}_s)$$

- Challenging to establish **well-posedness** (existence and uniqueness of solutions) (Mori & Ohm, Nonlinearity 2023)
- Challenging to solve numerically (du Roure et al., Annu. Rev. Fluid Mech. 2017)

Elastohydrodynamics equations

A "discrete" version: the N-link model



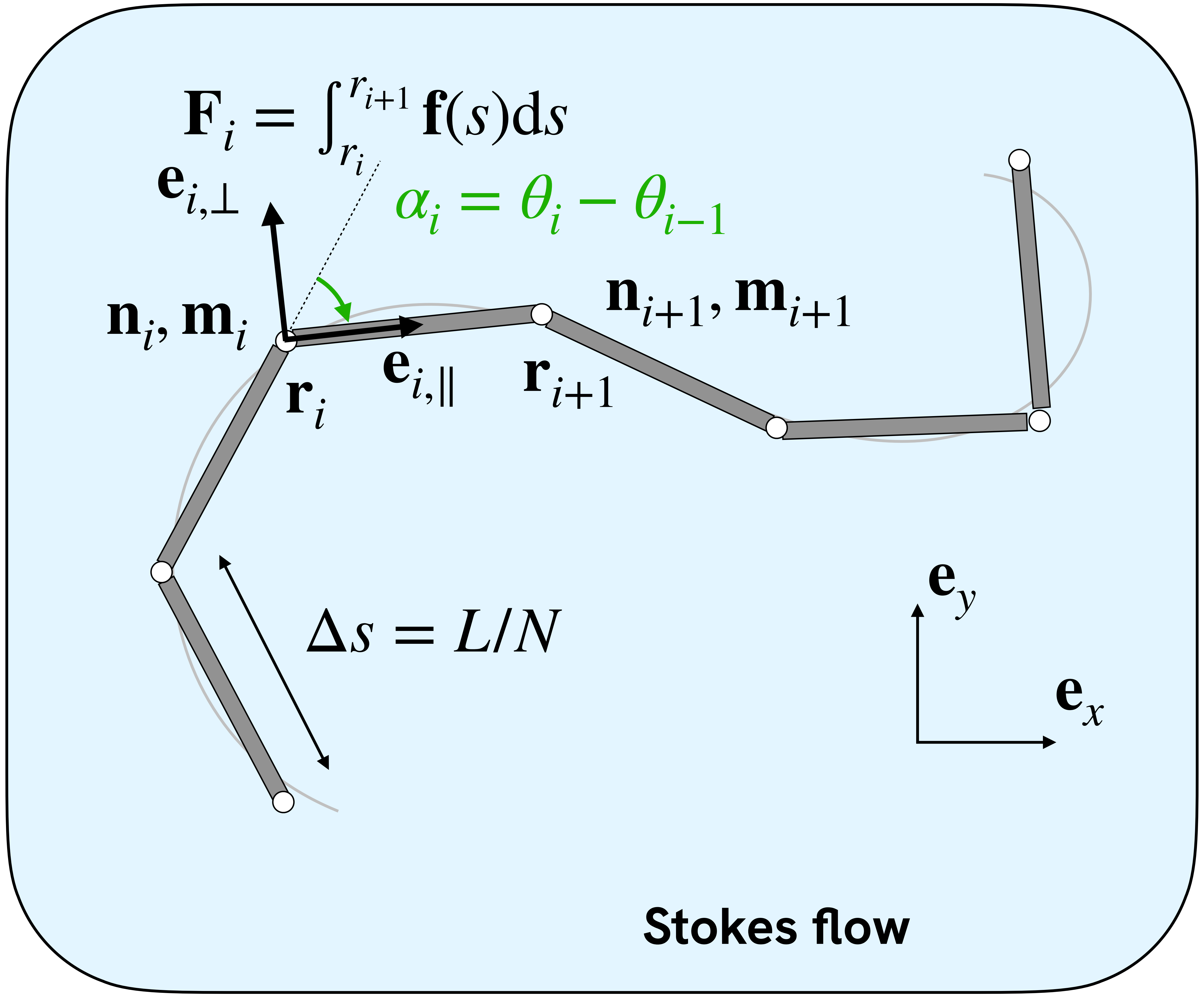


Alouges et al. 2015, Moreau et al. 2018

Elastohydrodynamics equations

A “discrete” version: the N-link model

- Approximate the filament with **rigid segments** connected by **elastic junctions**
- Integrate the **drag force density** over each segment → ODE system
- A “mechanical” discretisation: **not** a discretization of arc length !



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Elastohydrodynamics equations

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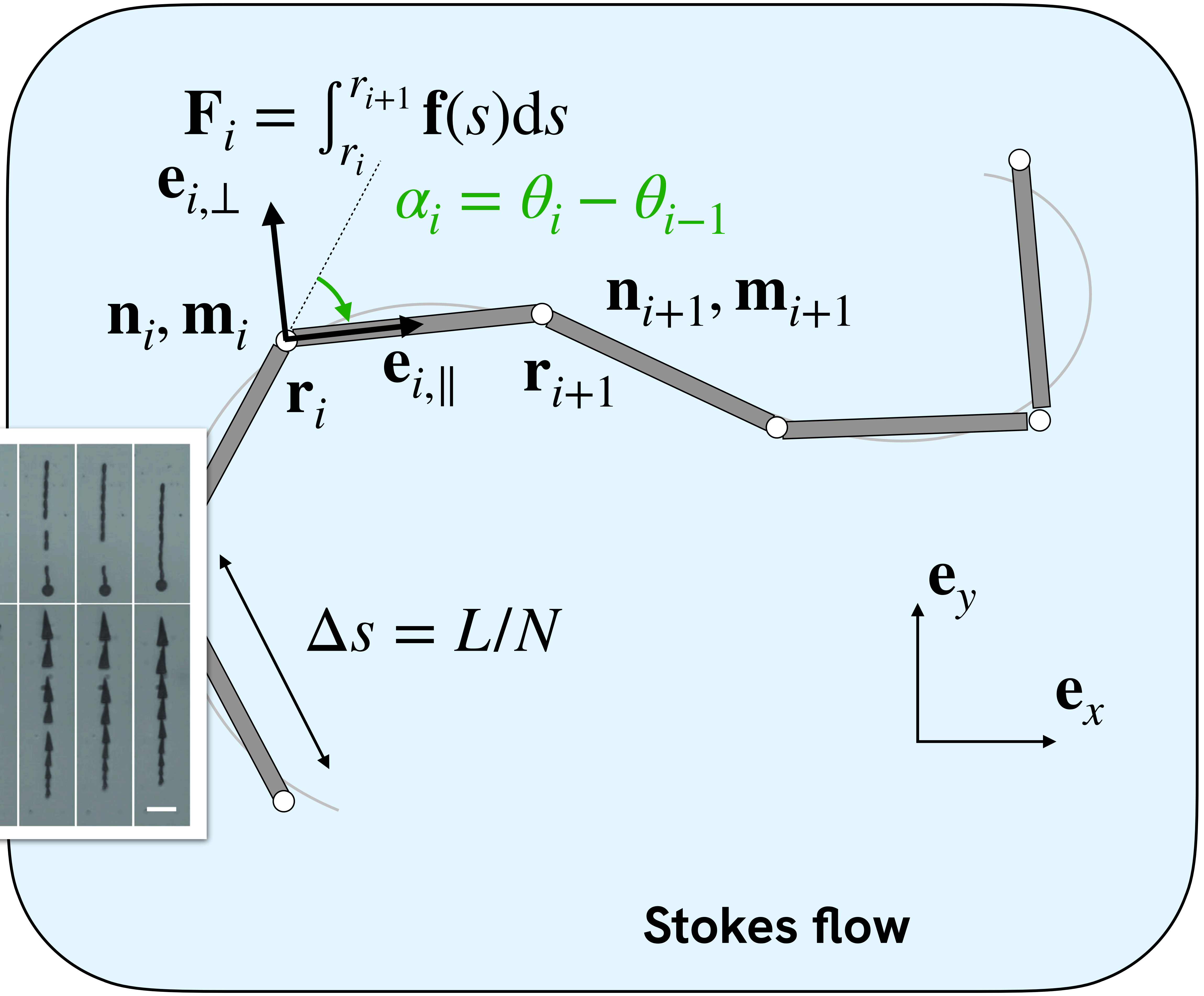
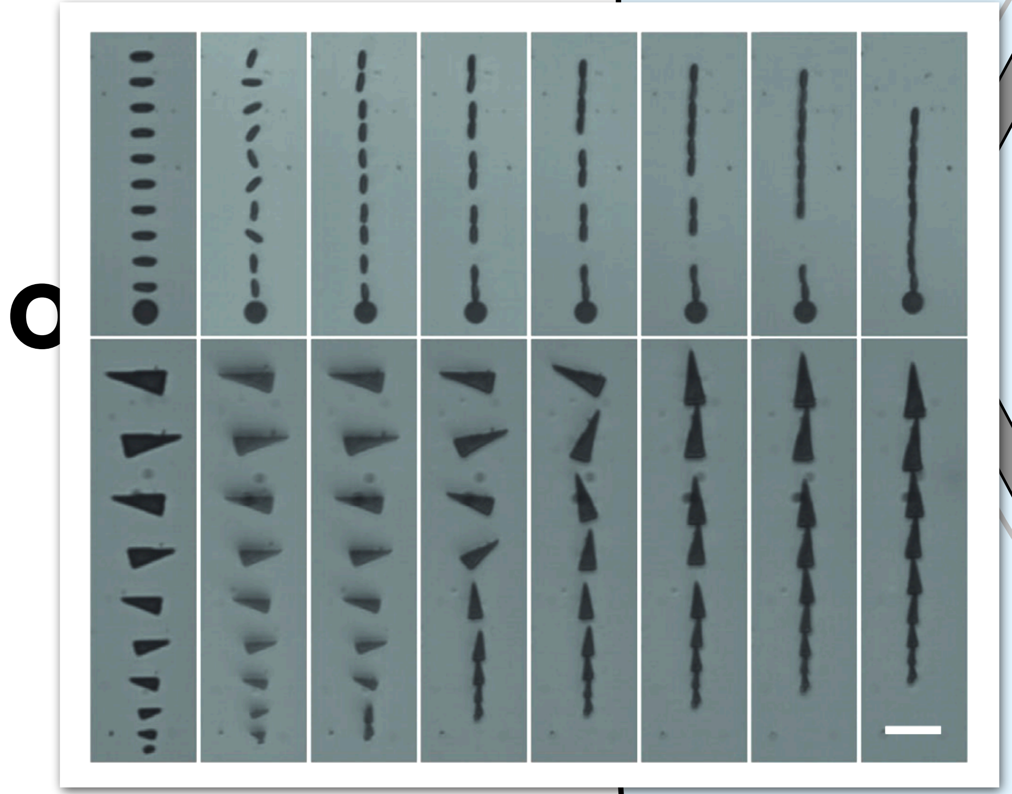
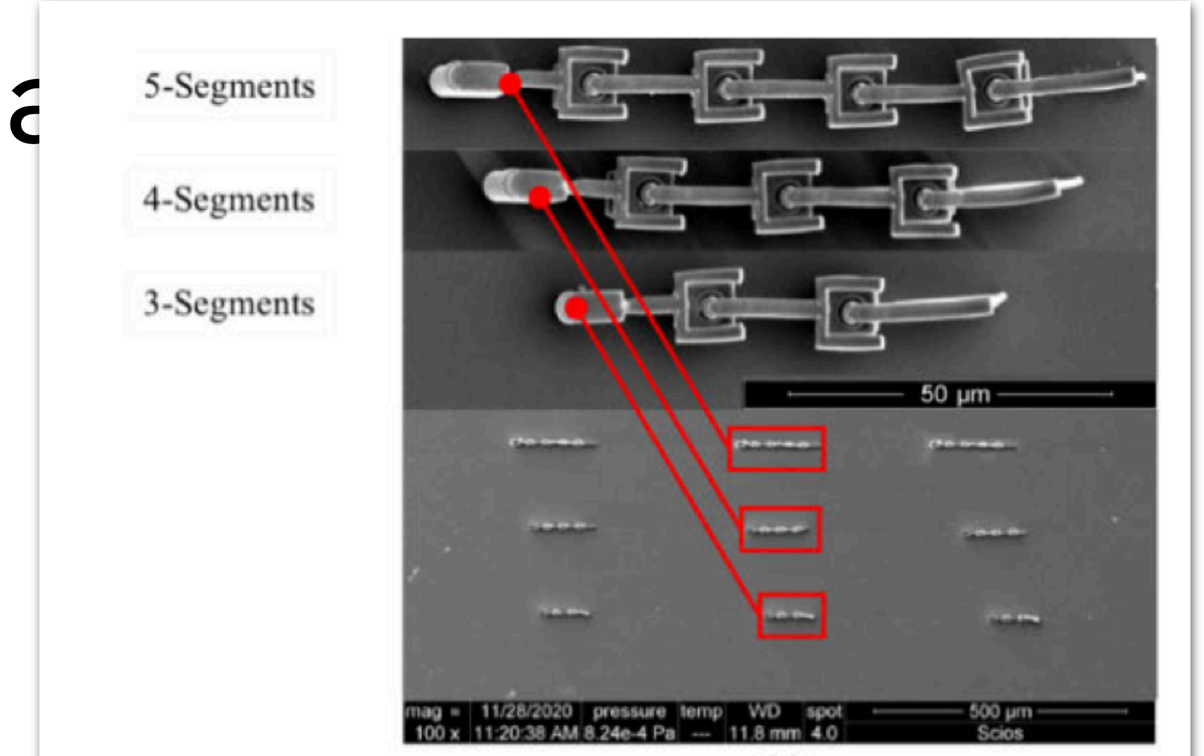
- Integrate the **drag force density** over each segment → ODE system
- A “mechanical” discretisation: **no**
discretiza







(e)

Alouges et al. 2015, Moreau et al. 2018

Elastohydrodynamics equations

A "discrete" version: the N-link model

$\forall i \in 1,\dots,N$

(Disc)

(local force and torque balance)

$$\mathbf{n}_{i+1} - \mathbf{n}_i + \Delta s\, \mathbf{C}(\theta_i)\, \dot{\mathbf{r}}_{i+1/2} = 0$$

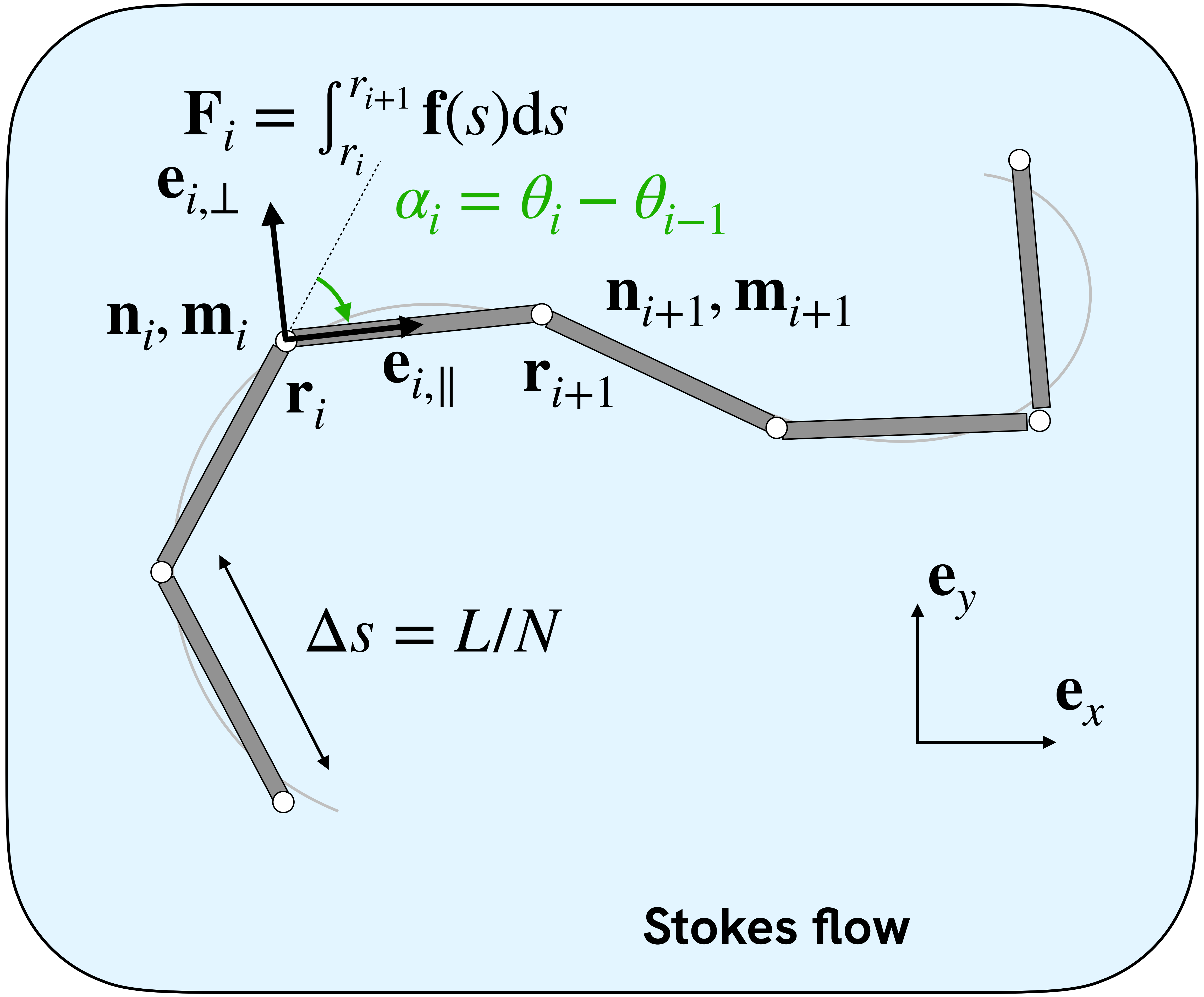
$$\mathbf{m}_{i+1} - \mathbf{m}_i + \Delta s\, \mathbf{e}_{i,\parallel} \times \frac{\mathbf{n}_{i+1} + \mathbf{n}_i}{2} - \frac{\Delta s^3}{12} c_\perp \dot{\theta}_i \mathbf{e}_z = 0$$

$$\mathbf{m}_{i+1} = \frac{E}{\Delta s}(\theta_{i+1} - \theta_i)\mathbf{e}_z$$ (elastic constitutive law)

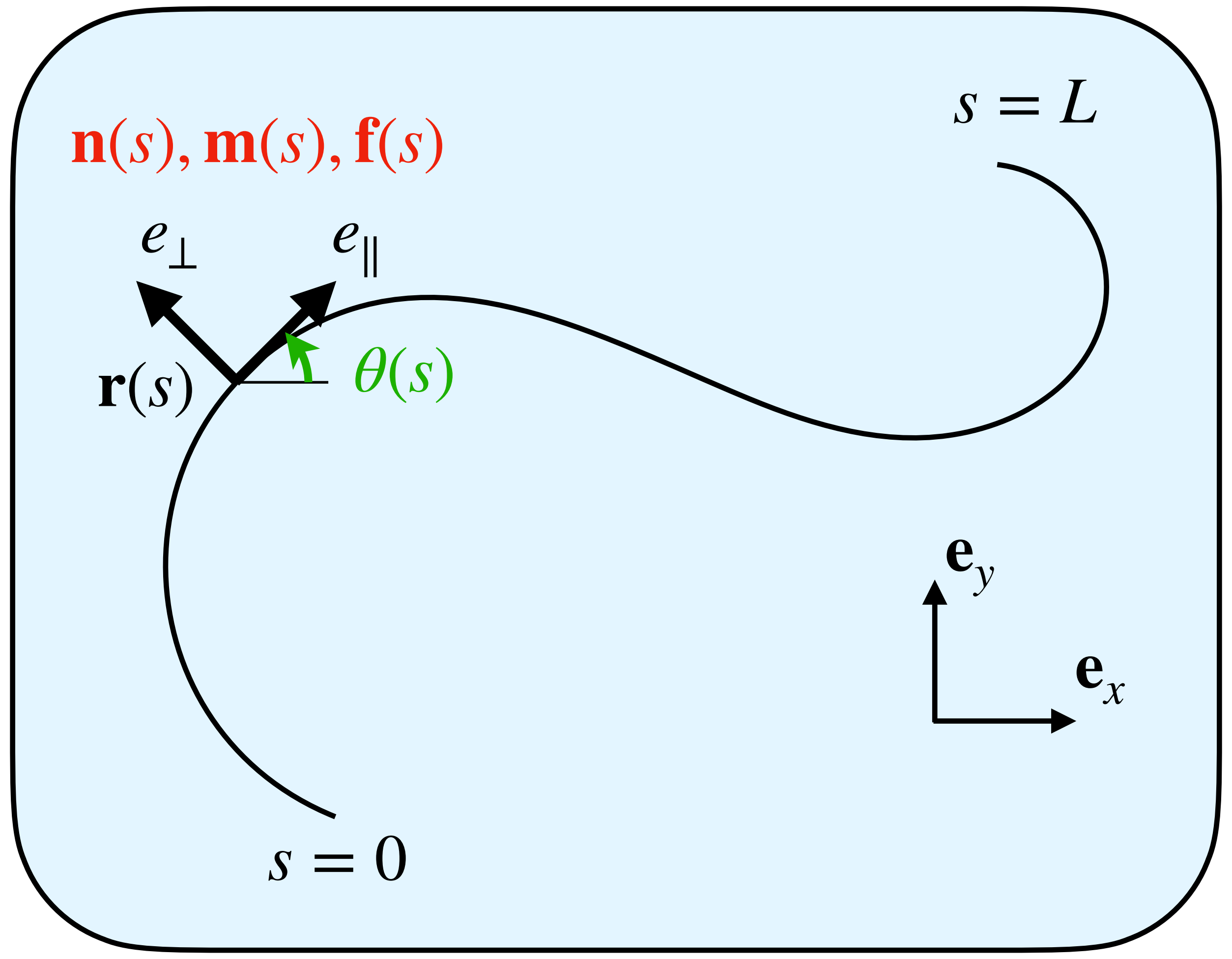
$$|\mathbf{r}_{i+1} - \mathbf{r}_i| = \Delta s$$ (inextensibility)

+ B.C.

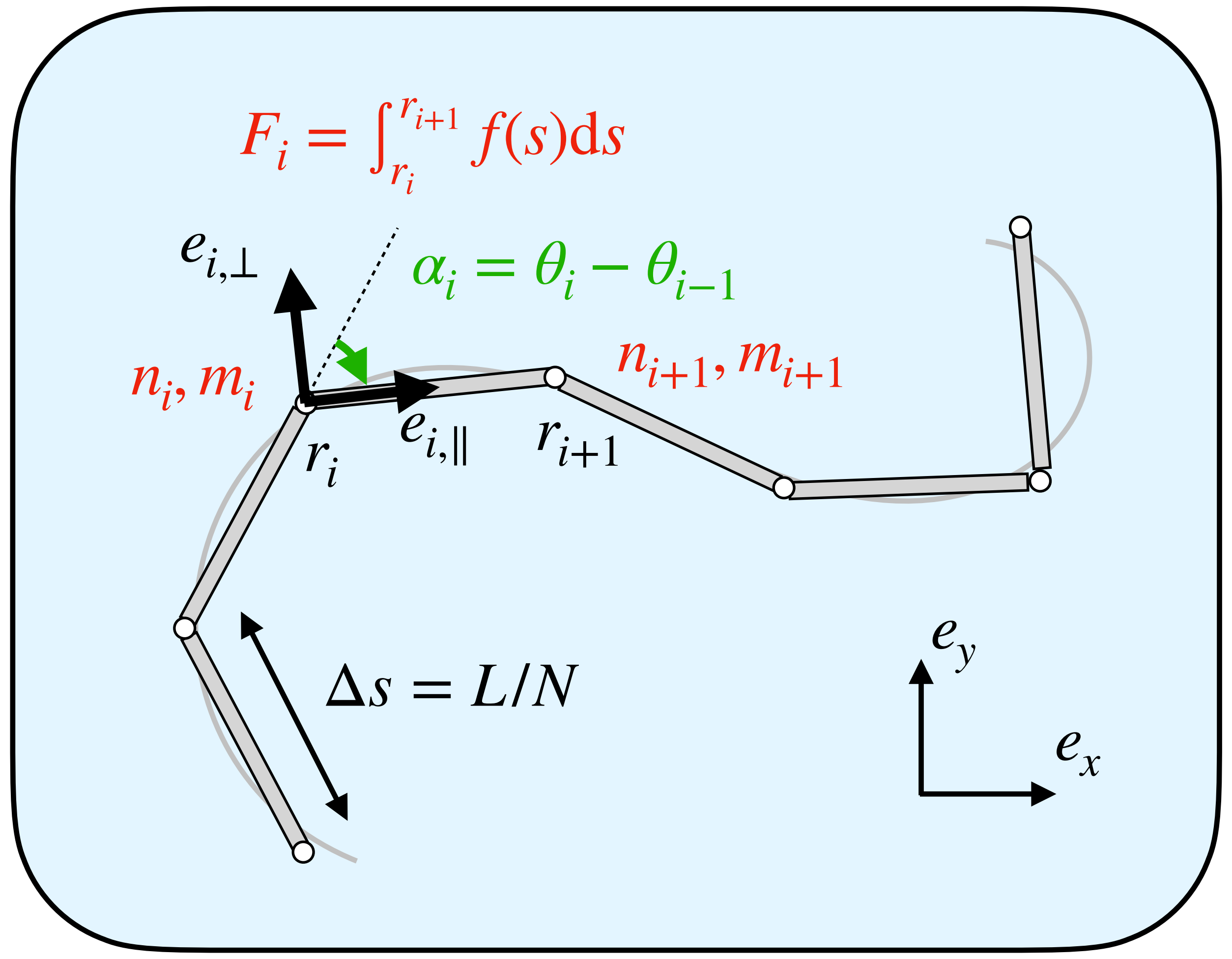
- **Finite-dimensional, more efficient and robust numerically**



Alouges et al. 2015, Moreau et al. 2018

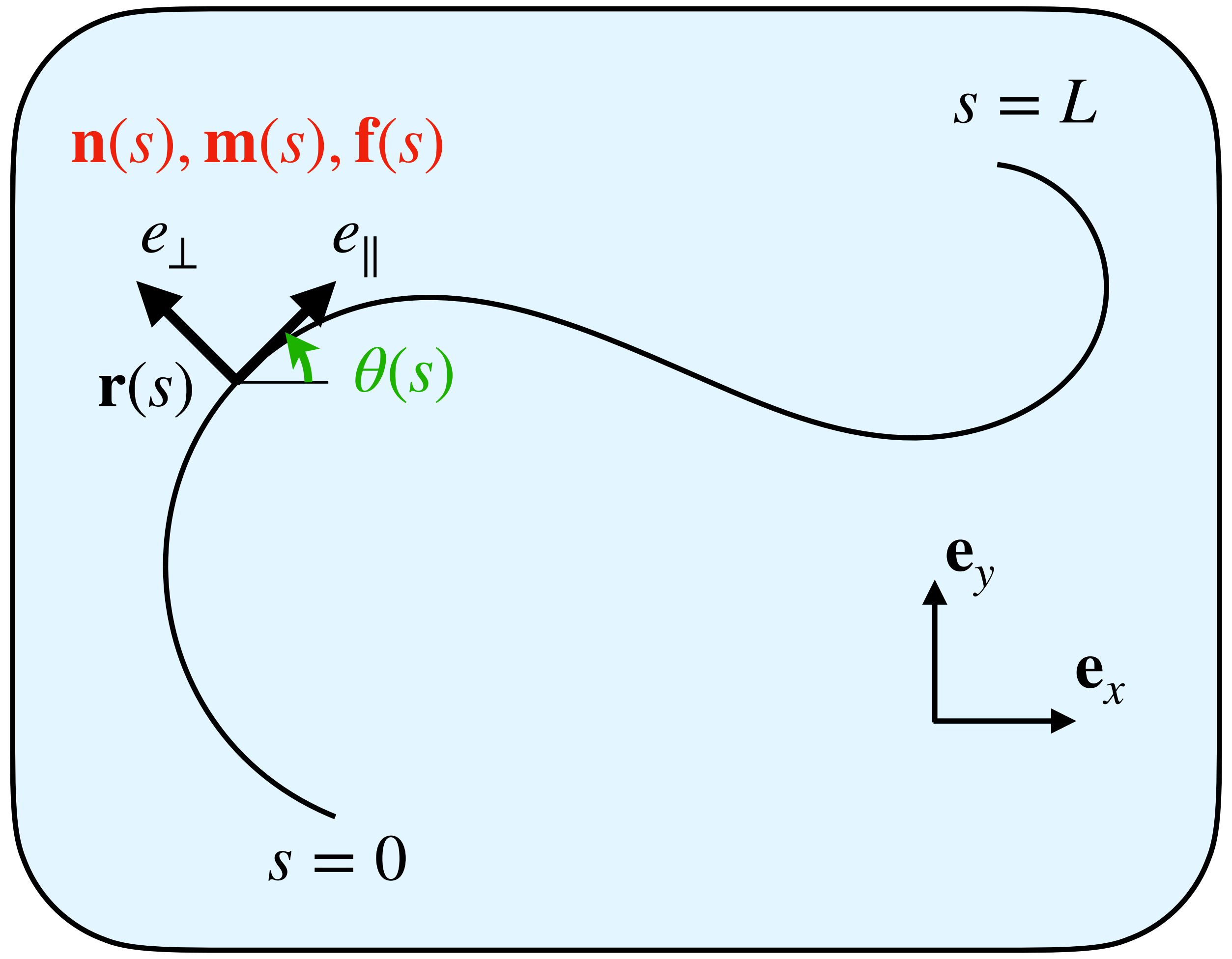


Comparing both models



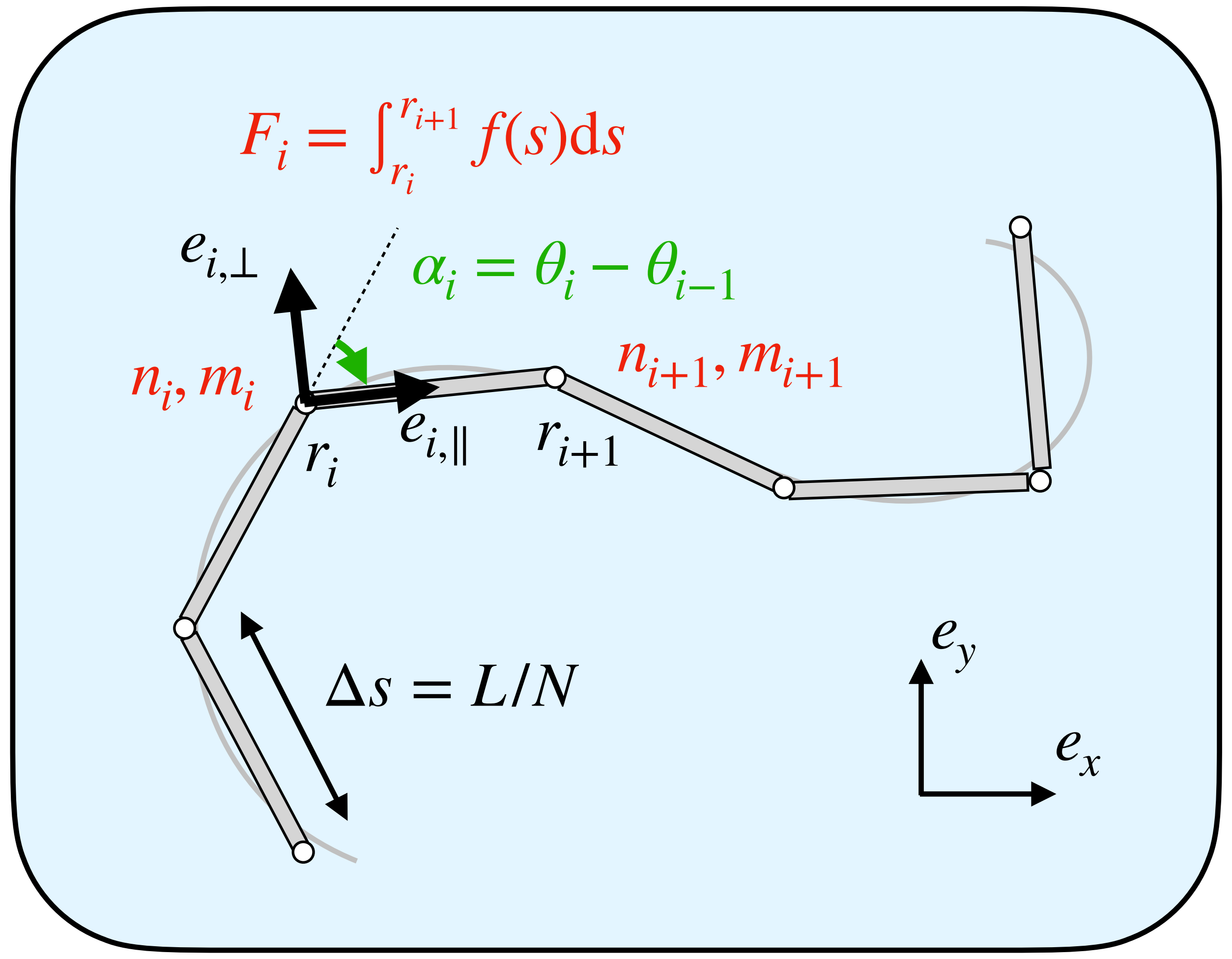
	(Cont)	(Disc)
Local force balance	$\partial_s \mathbf{n} + \mathbf{C}(\theta)\partial_t \mathbf{r} = 0$	$n_{i+1} - n_i + \Delta s\, \mathbf{C}(\theta_i)\, \dot{r}_{i+1/2} = 0$
Local torque balance	$\partial_s \mathbf{m} + \mathbf{r}_s \times \mathbf{n} = 0$	$m_{i+1} - m_i + \Delta s\, e_{i,\parallel} \times \frac{n_{i+1} + n_i}{2} - \frac{\Delta s^3}{12} c_\perp \dot{\theta}_i e_z = 0$
Elastic constitutive law	$\mathbf{m} = E\theta_s \mathbf{e}_z$	$m_{i+1} = \frac{E}{\Delta s}(\theta_{i+1} - \theta_i) e_z$
Inextensibility	$\lvert \mathbf{r}_s \rvert^2 = 1$	$\frac{\lvert r_{i+1} - r_i \rvert}{\Delta s} = 1$

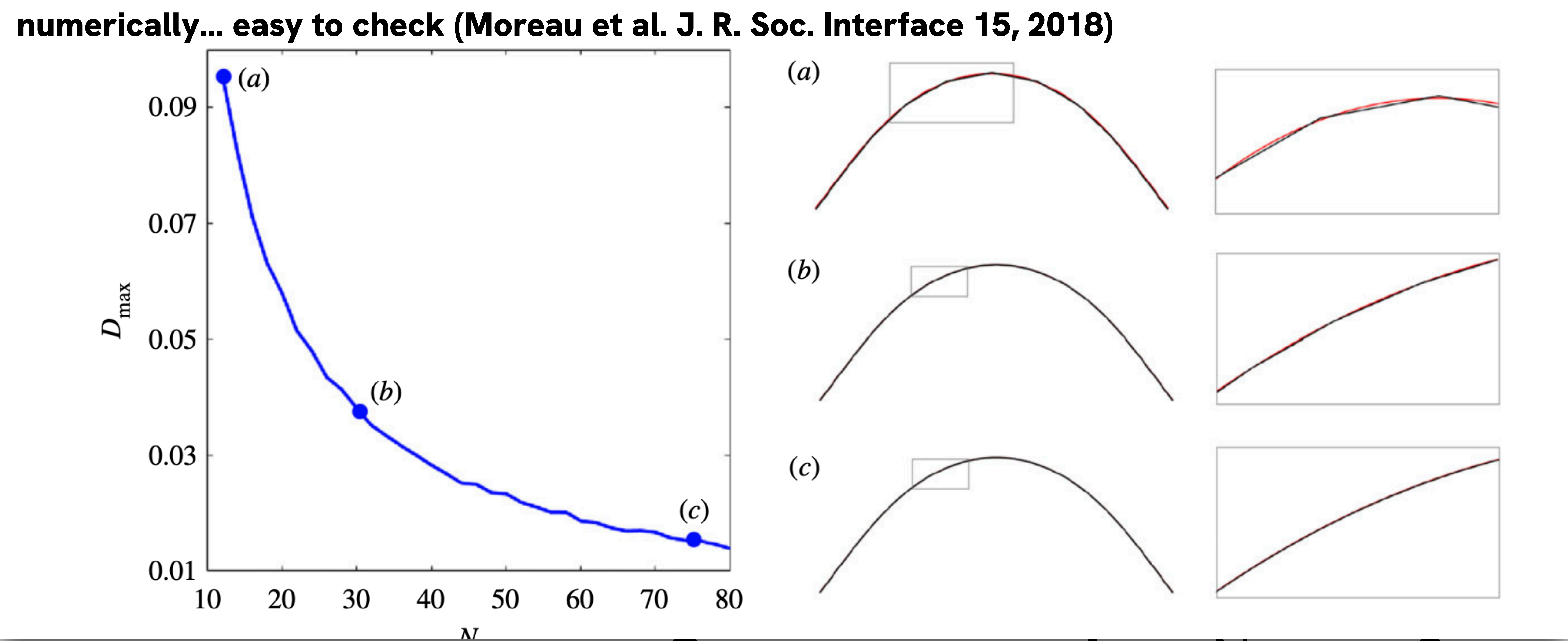
- **Convergence when $N \to \infty$?**



Comparing both models

(Cont) (Disc)





$\mathbf{C}(\theta_i)\, \dot{r}_{i+1/2} = 0$

$\frac{n_{i+1} + n_i}{2} - \frac{\Delta s^3}{12} c_\perp \dot{\theta}_i e_z = 0$

$_{i+1} - \theta_i) e_z$

$= 1$

- **Convergence when $N \to \infty$?**

A convergence result

Theorem 1 (Alouges, Lefebvre-Lepot, Levillain, M., arXiv:2502.09988)

$$\text{(Disc)} \xrightarrow[N \to \infty]{} \text{(Cont)}$$

A convergence result

Theorem 1

(Cont)

I.C. $\mathbf{r}^0$ in $C^2(0,L)$

A convergence result

Theorem 1

(Disc): $\big((\mathbf{r}_i)_i, (\mathbf{n}_i)_i, (\mathbf{m}_i)_i\big)$ — (Cont)

I.C. $(\mathbf{r}_i^0)_i$ ← define — I.C. $\mathbf{r}^0$ in $C^2(0,L)$

A convergence result

Interpolates

$$\mathbf{r}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{r}_i(t)\phi_i(s), \qquad m^h(t,s) = \sum_{i=1}^{N+1} m_i(t)\phi_i(s) \qquad \mathbf{n}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{n}_i(t)\phi_i(s)$$

Theorem 1

(Disc): $\left((\mathbf{r}_i)_i, (\mathbf{n}_i)_i, (\mathbf{m}_i)_i\right)$

I.C. $(\mathbf{r}_i^0)_i$

(Cont)

I.C. $\mathbf{r}^0$ in $C^2(0,L)$

A convergence result

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I.C. $(\mathbf{r}_i^0)_i$

(Cont)

I.C. $\mathbf{r}^0$ in $C^2(0,L)$

Assumptions (can always be satisfied)

- $\mathbf{r}^h(0,\cdot) \longrightarrow_{N\to\infty} \mathbf{r}^0 \quad \text{in} \quad L^2(0,L)$
- $\exists C^0 > 0, \frac{E}{2}\sum_{i=1}^{N-1} h\left(\frac{\theta_{i+1}(0)-\theta_i(0)}{h}\right)^2 = C_h^0 \leq C^0$

A convergence result

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Theorem 1

(Disc): $\left((\mathbf{r}_i)_i, (\mathbf{n}_i)_i, (\mathbf{m}_i)_i\right)$

I.C. $(\mathbf{r}_i^0)_i$

I.C. $\mathbf{r}^0$

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Then,

$\mathbf{r}^h \rightharpoonup \mathbf{r} \quad \text{in} \quad H^1(Q_T)$

$m^h \rightharpoonup m \quad \text{in} \quad L^2(0,T; H^1(0,L))$

$\mathbf{n}^h \rightharpoonup \mathbf{n} \quad \text{in} \quad L^2((0,T); H^1(0,L))$

A convergence result

Interpolates

$$\mathbf{r}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{r}_i(t)\phi_i(s), \qquad m^h(t,s) = \sum_{i=1}^{N+1} m_i(t)\phi_i(s) \qquad \mathbf{n}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{n}_i(t)\phi_i(s)$$

Theorem 1

(Disc): $\left((\mathbf{r}_i)_i, (\mathbf{n}_i)_i, (\mathbf{m}_i)_i\right)$

I.C. $(\mathbf{r}_i^0)_i$

I.C. $\mathbf{r}^0$

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Then,

$$\mathbf{r}^h \rightharpoonup \mathbf{r} \quad \text{in} \quad H^1(Q_T)$$

$$m^h \rightharpoonup m \quad \text{in} \quad L^2(0,T;H^1(0,L))$$

$$\mathbf{n}^h \rightharpoonup \mathbf{n} \quad \text{in} \quad L^2((0,T);H^1(0,L))$$

and $(\mathbf{r}, m, \mathbf{n})$ is weak sol. of (Cont)

Proof scheme

1. Show that $(\mathbf{r}^h, m^h, \mathbf{n}^h)$ are **bounded uniformly** in h in the relevant spaces

2. Extract a **weakly convergent subsequence** $(\mathbf{r}, m, \mathbf{n})$

$\mathbf{r}^h$	$H^1(Q_T)$
m^h	$L^2(0,T; H^1(0,L))$
$\mathbf{n}^h$	$L^2(0,T; H^1(0,L))$

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3. Show that the limits are (*weakly*) solution to (Cont)

tricky! we need **additional bounds**, and **strong** convergence for some of them

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$\mathbf{n}^h$	$L^2(0,T;H^1(0,L))$
$\mathbf{r}_s^h$	$L^2(0,T;BV(0,L))$
$\dot{\mathbf{r}}_s^h$	$L^2(0,T;H^{-1}(0,L))$
$h\dot{\bar{\theta}}^h$	$L^2(Q_T)$
θ_s^h	$L^2(Q_T)$

tricky! we need **additional bounds**, and **strong** convergence for some of them

Detail on point 3

for all test functions φ in $C_c^\infty(0,L)$, ψ in $C_c^\infty(0,T)$

$$\int_0^T \int_0^L \Big(\mathbf{C}(r_s(t,s))\dot{r}(t,s) + n_s(t,s)\Big)\,\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t = 0$$

force balance

$$\int_0^T \int_0^L \Big(m_s(t,s) + r_s(t,s) \times n(t,s)\Big)\,\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t = 0$$

moment balance

$$\int_0^T \int_0^L m^z(t,s)\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t = E\int_0^T \int_0^L (r_{ss}(t,s) \times r_s(t,s)) \cdot e_z\varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t$$

constitutive equation

Detail on point 3

$$\int_0^T \int_0^L \left(\mathbf{C}(r_s(t,s))\dot{r}(t,s) + n_s(t,s) \right) \varphi(s)\psi(t)\, \mathrm{d}s\, \mathrm{d}t = 0$$

RFT:
$$\mathbf{C}(\theta(t,s))\dot{\mathbf{r}} = -c_{\parallel}(\mathbf{e}_{\parallel} \cdot \dot{\mathbf{r}})\mathbf{e}_{\parallel} - c_{\perp}(\mathbf{e}_{\perp} \cdot \dot{\mathbf{r}})\mathbf{e}_{\perp}$$

- morally, product of $\mathbf{r}_s^2$ and $\dot{\mathbf{r}}$
- for weak convergence of a product $f \times g$, we need **strong** convergence of f and **weak** convergence of g
- so, we need **strong** convergence of $\mathbf{r}_s^2$ in $L^2 \to$ **strong** convergence of $\mathbf{r}_s$ in L^4

A BV estimate

$$\int_0^T \int_0^L \left(\mathbf{C}(r_s(t,s))\dot{r}(t,s) + n_s(t,s) \right) \varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t = 0$$

RFT:

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Theorem (Aubin 1963)

- Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$.
- Suppose that $X_0 \Subset X$ and $X \hookrightarrow X_1$.
- Let $1 \leqslant p, q < +\infty$, let $W = \{u \in L^p(0,T;X_0) \mid \dot{u} \in L^q(0,T;X_1)\}$.
- Then the embedding of W into $L^p(0,T;X)$ is compact.

A BV estimate

$$\int_0^T \int_0^L \left(\mathbf{C}(r_s(t,s))\dot{r}(t,s) + n_s(t,s) \right) \varphi(s)\psi(t)\,\mathrm{d}s\,\mathrm{d}t = 0$$

RFT:

$$\mathbf{C}(\theta(t,s))\dot{\mathbf{r}} = -c_{\parallel}(\mathbf{e}_{\parallel} \cdot \dot{\mathbf{r}})\mathbf{e}_{\parallel} - c_{\perp}(\mathbf{e}_{\perp} \cdot \dot{\mathbf{r}})\mathbf{e}_{\perp}$$

Theorem (Aubin 1963)

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$H^{-1}(0,L)$

$\Cap$ compact embedding

$L^4(0,L)$

$\hookrightarrow$ continuous embedding

$BV(0,L)$

Key formula: energy dissipation for (Disc)

Proposition:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[E\sum_{i=1}^{N-1}\Delta s\left(\frac{\theta_{i+1}-\theta_i}{\Delta s}\right)^2\right]+\sum_{i=1}^{N}\frac{\Delta s^3}{12}c_\perp\dot{\theta}_i^2-\sum_{i=1}^{N}\Delta s\,\mathbf{C}(\theta_i)\,\dot{r}_{i+\frac{1}{2}}\cdot\dot{r}_{i+\frac{1}{2}}=0.$$

Elastic energy

Hydrodynamic drag dissipation

Bounds

$\mathbf{r}^h$	$H^1(Q_T)$	inextensibility + energy dissipation + Poincaré-Wirtinger
m^h	$L^2(0,T;H^1(0,L))$	energy dissipation
$\mathbf{n}^h$	$L^2(0,T;H^1(0,L))$	energy dissipation
$\mathbf{r}_s^h$	$L^2(0,T;BV(0,L))$	compute total variation + assumption on bending energy
$\dot{\mathbf{r}}_s^h$	$L^2(0,T;H^{-1}(0,L))$	consequence of (1)
$h\dot{\bar{\theta}}^h$	$L^2(Q_T)$	energy dissipation
θ_s^h	$L^2(Q_T)$	energy dissipation

Convergences

$\mathbf{r}^h$	$\mathbf{r}$	$H^1(Q_T)$	weak	immediate from boundedness
$\mathbf{r}^h$	$\mathbf{r}$	$L^2(Q_T)$	**strong**	Rellich-Kondrachov theorem
$\mathbf{r}_s^h$	$\mathbf{r}_s$	$L^2(0,T; L^p(0,L))$	**strong**	compact embedding of BV + Aubin-Lions-Simon theorem
$\hat{\mathbf{r}}^h$	$\mathbf{r}$	$L^2(0,T; H^1(0,L))$	**strong**	connect to bounds on m and θ
$\hat{\mathbf{r}}^h$	$\mathbf{r}$	$L^2(0,T; H^2(0,L))$	weak	
$\mathbf{n}^h$	$\mathbf{n}$	$L^2(0,T; H^1(0,L))$	weak	immediate from boundedness
$\mathbf{m}^h$	$\mathbf{m}$	$L^2(0,T; H^1(0,L))$	weak	immediate from boundedness
$\bar{\mathbf{m}}^h$	$\mathbf{m}$	$L^2(0,T; L^p(0,L))$	weak	show that $\Vert\mathbf{m} - \bar{\mathbf{m}}\Vert_{L^2(Q_T)} \to 0$
$h^2\dot{\bar{\theta}}^h$	0	$L^2(Q_T)$	**strong**	immediate from boundedness
θ_s^h	α	$L^2(Q_T)$	weak	immediate from boundedness

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$\square$

Back to the theorem

Interpolates

$$\mathbf{r}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{r}_i(t)\phi_i(s), \qquad m^h(t,s) = \sum_{i=1}^{N+1} m_i(t)\phi_i(s) \qquad \mathbf{n}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{n}_i(t)\phi_i(s)$$

Theorem 1 (Alouges, Lefebvre-Lepot, Levillain, M., arXiv:2502.09988)

(Disc): $\left((\mathbf{r}_i)_i, (\mathbf{n}_i)_i, (\mathbf{m}_i)_i\right)$

I.C. $(\mathbf{r}_i^0)_i$

I.C. $\mathbf{r}^0$

Assumptions (can always be satisfied)

- $\mathbf{r}^h(0,\cdot) \longrightarrow_{N\to\infty} \mathbf{r}^0 \quad \text{in} \quad L^2(0,L)$
- $\exists C^0 > 0, \frac{E}{2}\sum_{i=1}^{N-1} h \left(\frac{\theta_{i+1}(0) - \theta_i(0)}{h}\right)^2 = C_h^0 \leq C^0$

Then,

$$\mathbf{r}^h \rightharpoonup \mathbf{r} \quad \text{in} \quad H^1(Q_T)$$
$$m^h \rightharpoonup m \quad \text{in} \quad L^2(0,T;H^1(0,L))$$
$$\mathbf{n}^h \rightharpoonup \mathbf{n} \quad \text{in} \quad L^2((0,T);H^1(0,L))$$

and $(\mathbf{r}, m, \mathbf{n})$ is sol. of (Cont)

Existence?

Interpolates

$$\mathbf{r}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{r}_i(t)\phi_i(s), \qquad m^h(t,s) = \sum_{i=1}^{N+1} m_i(t)\phi_i(s) \qquad \mathbf{n}^h(t,s) = \sum_{i=1}^{N+1} \mathbf{n}_i(t)\phi_i(s)$$

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Existence of solutions to (Disc)?

(Disc): $\big((\mathbf{r}_i)_i, (\mathbf{n}_i)_i, (\mathbf{m}_i)_i\big)$

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Theorem 2 (*Well-posedness of (Disc)*)

For all initial data, there exists a unique global solution of (Disc) in $C^1(\mathbb{R}_+)$.

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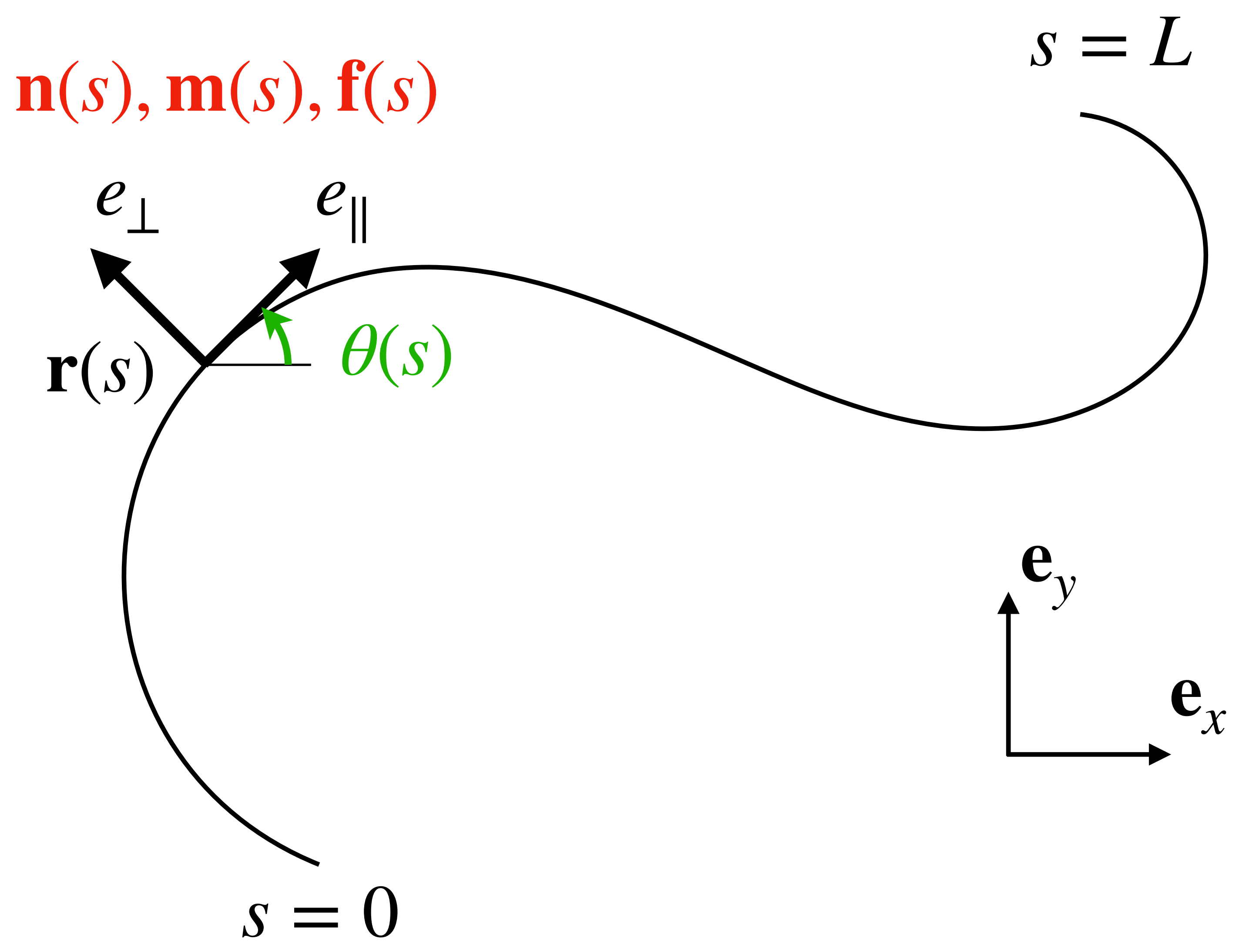
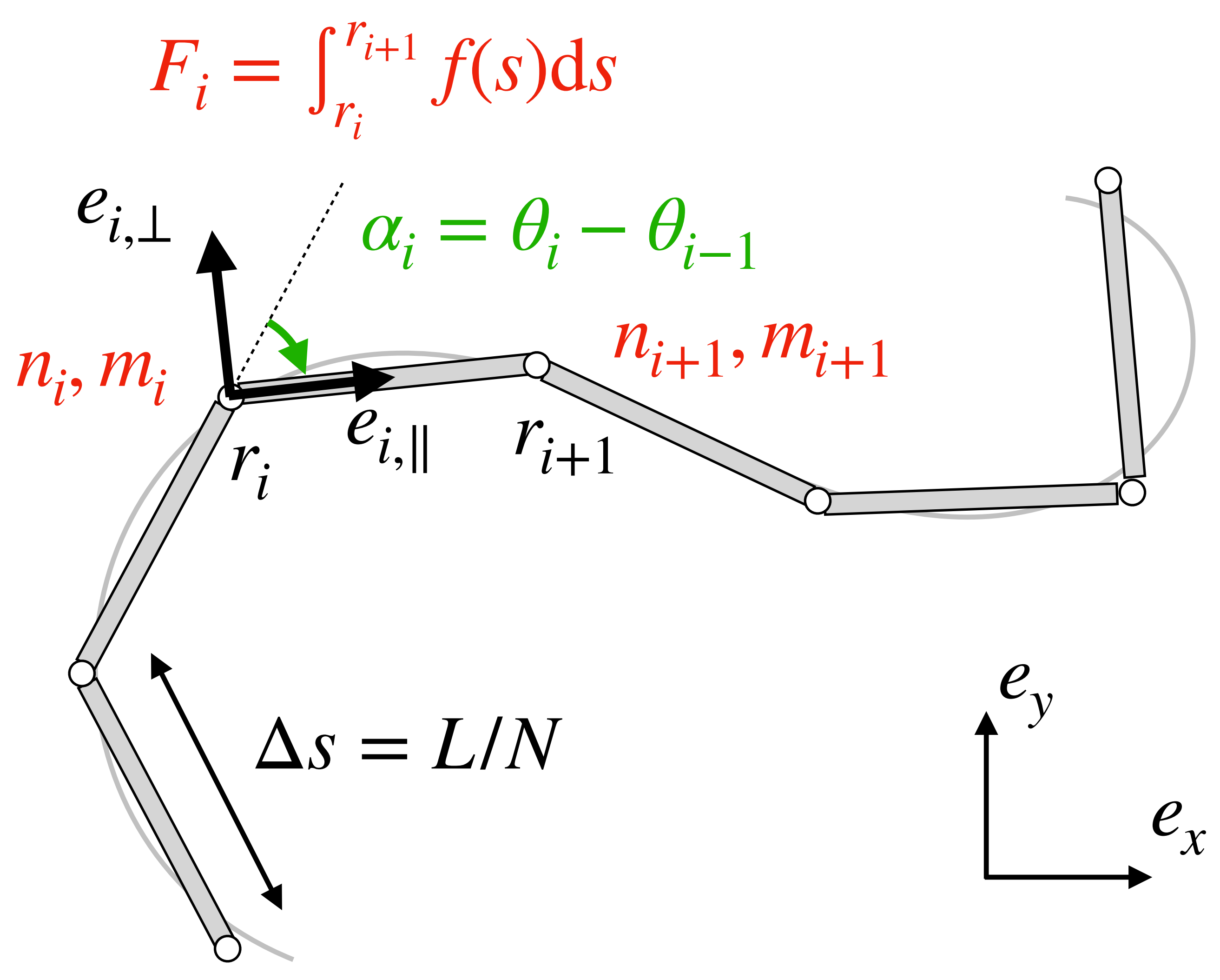
Theorem 2 (*Well-posedness of (Disc)*)

For all initial data, there exists a unique global solution of (Disc) in $C^1(\mathbb{R}_+)$.

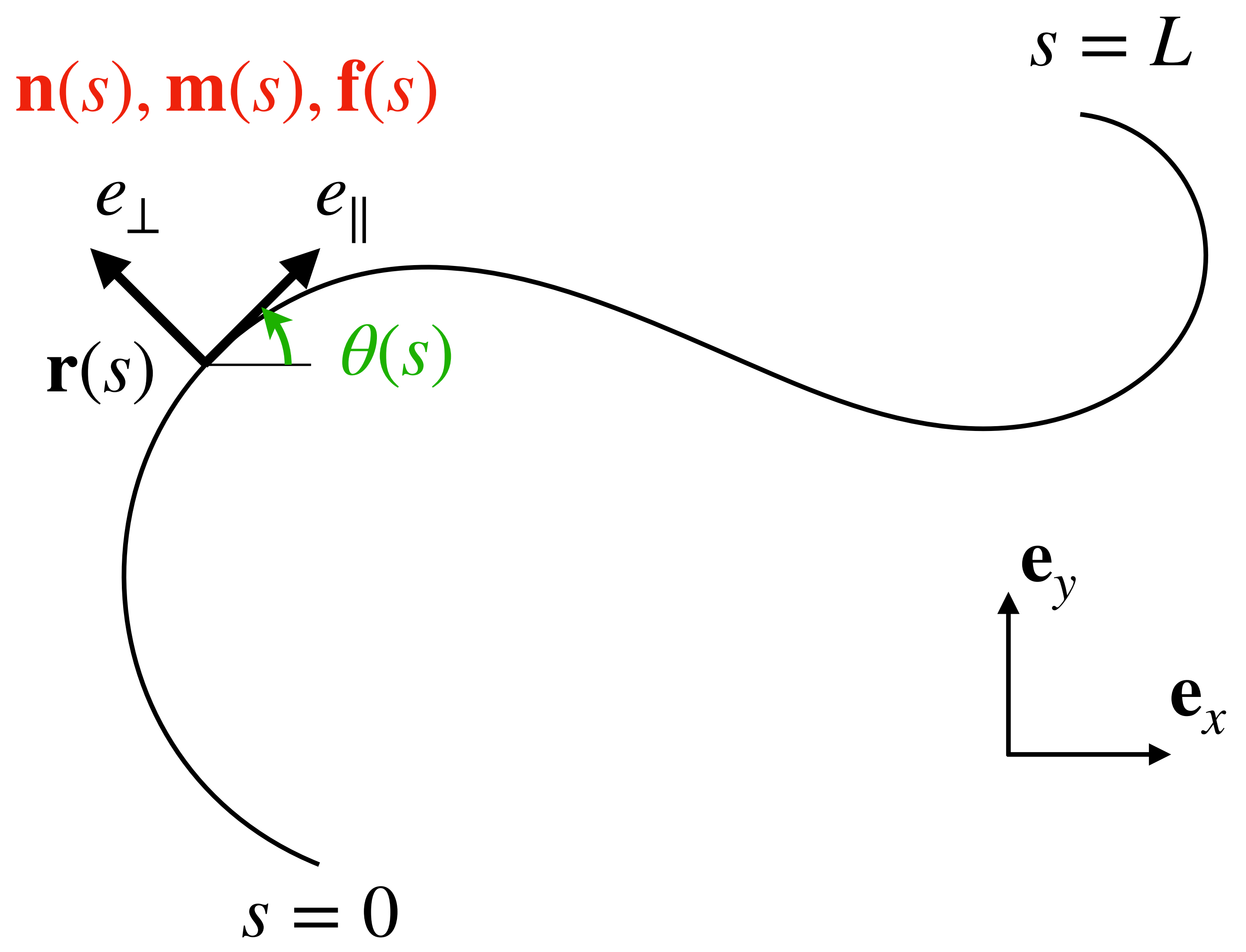
- Differential algebraic system: check invertibility + bounds to have global existence through Cauchy-Lipschitz theorem

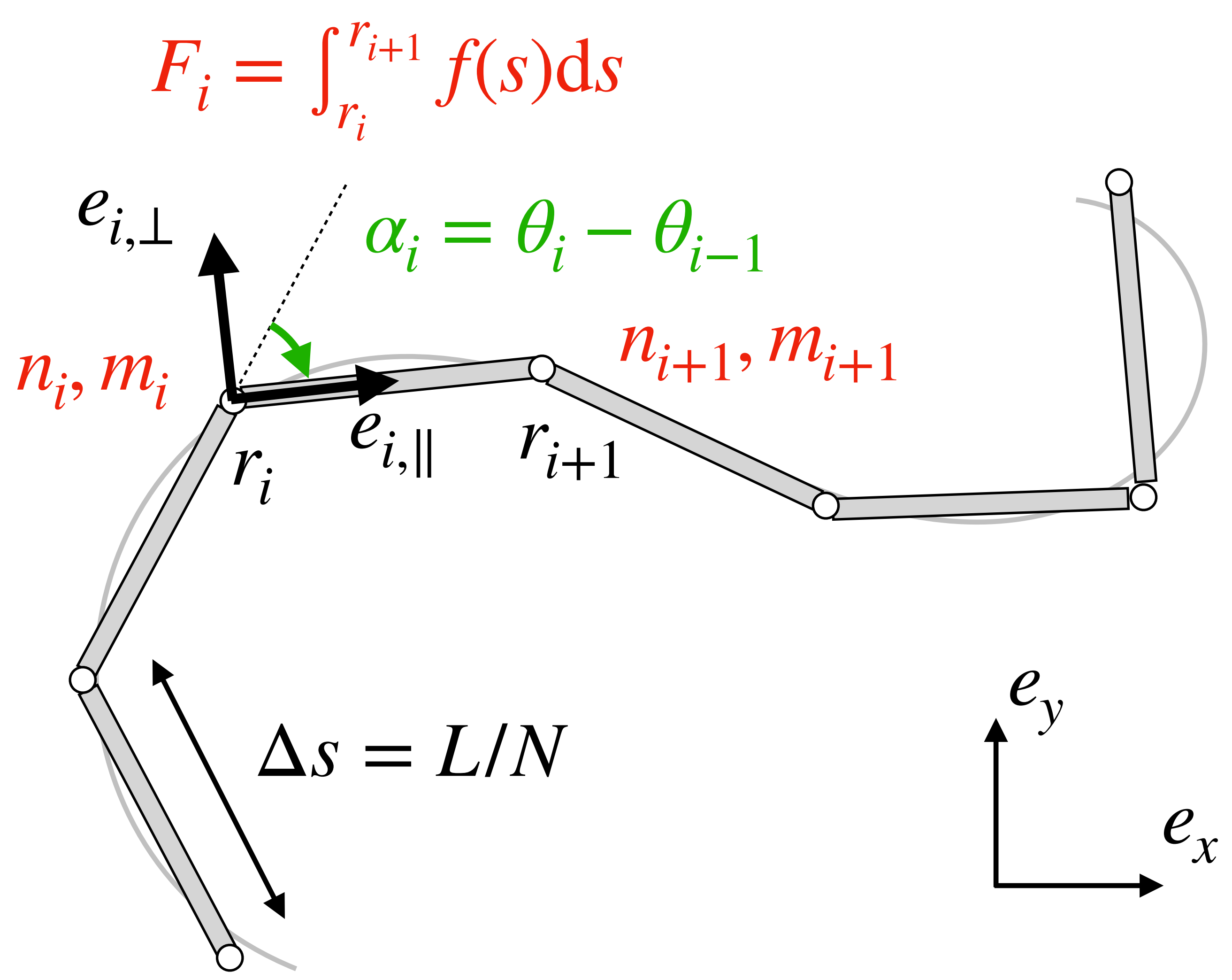




Theorem 1: Weak convergence of (Disc) towards (Cont)

Theorem 2: Well-posedness of the N-link model





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Well-posedness of (Cont)?

Well-posedness of (Cont)?

Theorem 1: Weak convergence of (Disc) towards (Cont)

Theorem 2: Existence and uniqueness of solutions for the N-link model

Corollary: Existence of solutions for the continuous model

but… up to the extraction of subsequences! → uniqueness is not given for free

Well-posedness of (Cont)?

Theorem 1: Weak convergence of (Disc) towards (Cont)

Theorem 2: Existence and uniqueness of solutions for the N-link model

Corollary: Existence of solutions for the continuous model in $H^1([0,T];H^1) \cap L^2([0,T],H^2)$

Theorem: (Mori, Ohm 2023) Existence and uniqueness of solutions *globally in time for small initial data in* $C^0([0,T];L^2) \cap C^0((0,T];H^1)$

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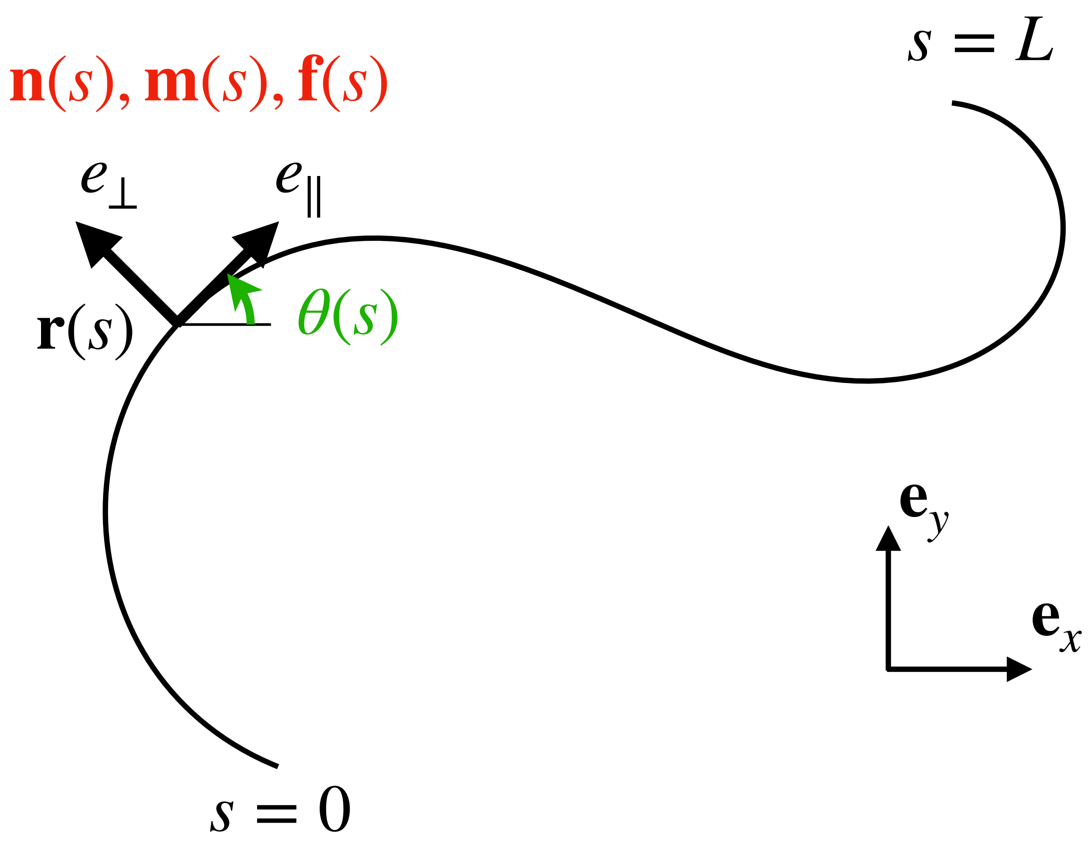
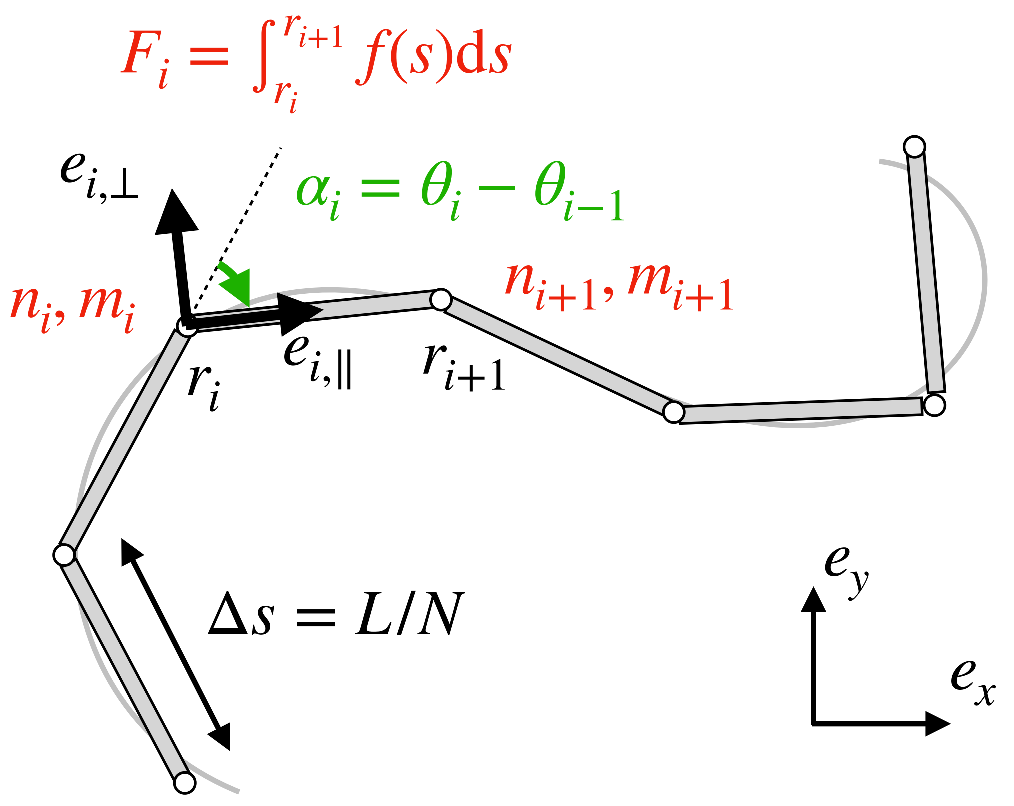
Theorem: (Albritton, Ohm 2025) Global existence and uniqueness of solutions *in* $L^2([0,T];H^2)$

Summary and outlook

arXiv:2502.09988





Theorem 1: Weak convergence of (Disc) towards (Cont)

Theorem 2: Well-posedness of the N-link model

Corollary: Existence of solutions for the continuous model

Extensions

- Internal activity, curvature at rest
- Refined hydrodynamics
- 3D, non-Newtonian fluids, multiple filaments...

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