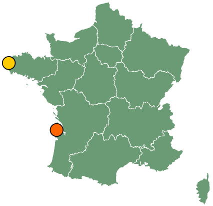

Stability of the trivial equilibrium in degenerate monostable reaction-diffusion equations

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Content

- Persistence *vs.* extinction for
  $\partial_t u = \Delta u + u^{1+p}(1-u)$ in $\mathbb{R}^N$

- Scalar Fujita-type results
  $\partial_t u = \Delta u + u^{1+p}(1-u)$ in $\mathbb{R}^N$

- Vectorial Fujita-type results — diffusion coupled systems

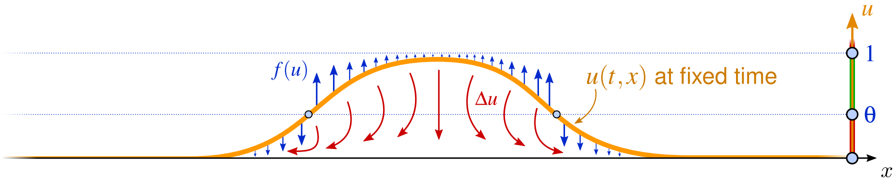
$$\begin{cases} \partial_t u = \Delta u - u + v + u^{1+p}(1-u) & \text{in } \mathbb{R}^N \\ \partial_t v = \Delta v + u - v + u^{1+q}(1-v) & \text{in } \mathbb{R}^N \end{cases}$$

- Further $\underline{\text{(intricate)}}$ diffusion processes

  fast diffusion channels, adiabatic moving boundaries...

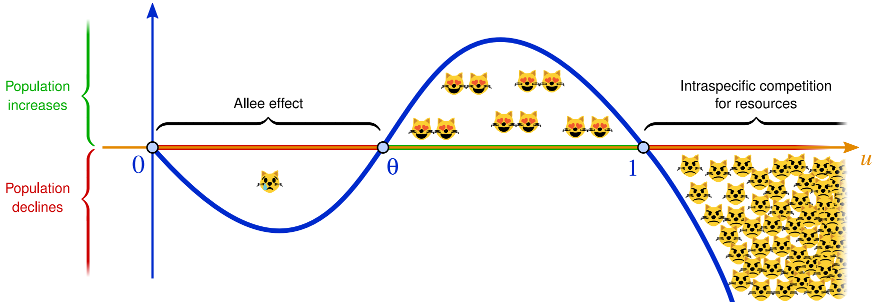
$u = u(t,x)$ represents the density of a population

$$\begin{cases} \partial_t u = \Delta u + f(u), & t > 0, \quad x \in \mathbb{R}^N, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^N. \end{cases}$$



Bistable : $f(u) = u(1-u)(u-\theta)$

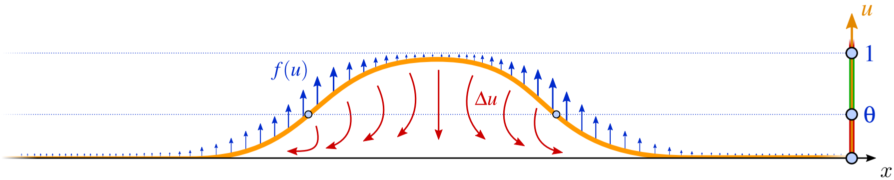
Strong Allee effect (low densities lead to negative growth)



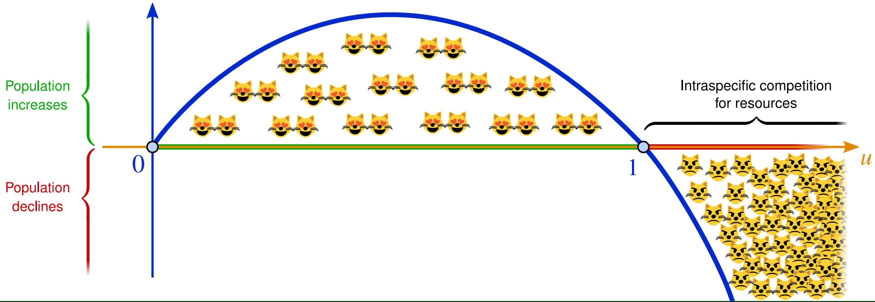
Reaction-diffusion equations

$u = u(t,x)$ represents the density of a population

$$\begin{cases} \partial_t u = \Delta u + f(u), & t > 0, \quad x \in \mathbb{R}^N, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^N. \end{cases}$$

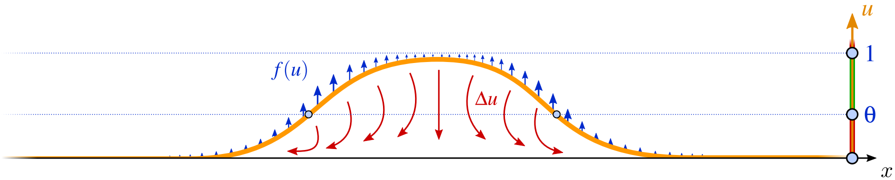


Fisher-KPP : $f(u) = u(1-u)$ **No Allee effect (individual growth rate is maximized as $u \to 0$)**



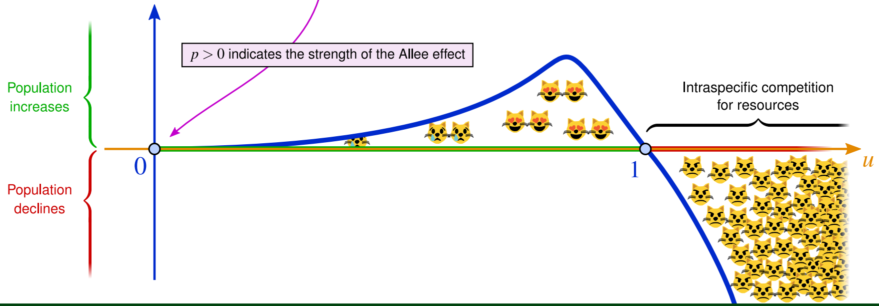
$u = u(t,x)$ represents the density of a population

$$\begin{cases} \partial_t u = \Delta u + f(u), & t > 0, \quad x \in \mathbb{R}^N, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^N. \end{cases}$$



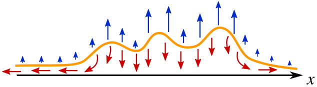
Degenerate monostable: $f(u) = u^{1+p}(1-u)$

Weak Allee effect (harder to grow at low densities, but still positive growth)



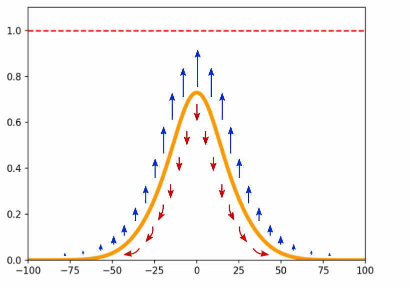
$$\partial_t u = \Delta u + u^{1+p}(1-u), \qquad t > 0, \quad x \in \mathbb{R}.$$

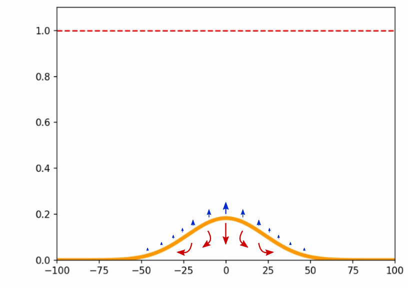
- Persistence or extinction? competition between growth and diffusion



$p \leq 2$ $\Big\}$ $\Rightarrow$ invasion

$p > 2$ and $u|_{t=0}$ "small" $\Big\}$ $\Rightarrow$ extinction





- See [Aronson and Weinberger, 1978] [Quittner and Souplet 2019]

Theorem (Aronson and Weinberger, 1978)

Consider $(\star)$: $\partial_t u = \Delta u + u^{1+p}(1-u) \qquad t > 0 \qquad x \in \mathbb{R}^N$

$1)$ $p \leq p_F := 2/N \Rightarrow$ systematic invasion (remind $p = 0$ is KPP)

$2)$ $p > p_F \Rightarrow$ possible extinction

- Partial answer to know whether the solutions to $(\star)$ persist or go extinct
- Comparison principle with the underlying ODE $U' = U^{1+p}(1-U)$ does not help since

$$U(t=0) > 0 \qquad \Rightarrow \qquad U(t) \overset{t\to\infty}{\to} 1$$

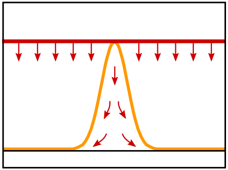
- When $p > p_F$, “smallness” of the initial data u_0 is determinant*
  - $2)$ is a straight consequence of Fujita’s theorem (see next slide)
  - $1)$ is more subtle and lies on the construction of appropriate subsolution
    see the book [Quittner, Souplet 2019] for an accessible proof

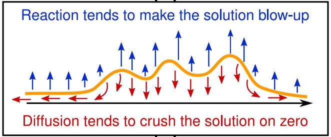
*mixture of L^1/L^∞ norms and fragmentation (defining fragmentation is quite difficult [Alfaro, Hamel, Roques (2024)] and has potentially a non-monotone effect [Garnier, Roques, Hamel (2012)])

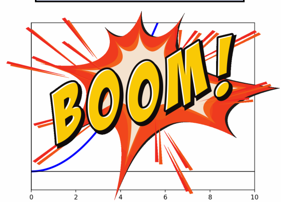
Fujita blow-up phenomena

$(\star)\ \partial_t u = \Delta u + u^{1+p}, \qquad t > 0,\ \ x \in \mathbb{R}^N$

Diffusion part	VS	Reaction part
$\partial_t u = \Delta u$		$U' = U^{1+p}$







$\|u(t,\cdot)\|_{L^\infty} \leq c/t^{N/2}$

Vanish with algebraic rate $N/2$

Global existence
or
blow-up in finite time ?

$U(t) = c/(T_{\text{boom}} - t)^{1/p}$

Blows-up with algebraic rate $1/p$

Theorem (H.Fujita) (1966)

If $\frac{1}{p} > \frac{N}{2}$, then **reaction always prevails over diffusion** : any positive solution to $(\star)$ blows-up in finite time.

If $\frac{1}{p} < \frac{N}{2}$, then **diffusion may prevail over reaction** : $(\star)$ admits global positive solutions.

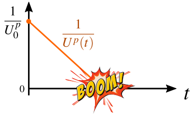
$$(\star)\ \partial_t u = \Delta u + u^{1+p}, \qquad t>0,\ \ x\in\mathbb{R}^N$$

Theorem (H.Fujita) (1966)

If $\frac{1}{p} > \frac{N}{2}$, then **reaction always prevails over diffusion** : any positive solution to $(\star)$ blows-up in finite time.

If $\frac{1}{p} < \frac{N}{2}$, then **diffusion may prevail over reaction** : $(\star)$ admits global positive solutions.

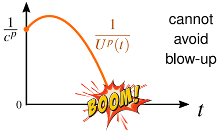
$$\boxed{U' = U^{1+p}} \xrightarrow{\text{solve}} \boxed{\frac{1}{U^p(t)} = \frac{1}{U_0^p} - pt}$$



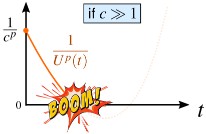
- Introduce time dependency on U_0 : $\quad U_0 = U_0(t) = c/(1+t)^{N/2}$ As for the solutions to the Heat equation!

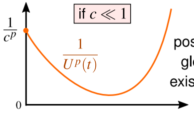
- This penalizes the growth of U: $\quad \dfrac{1}{U^p(t)} = \dfrac{(1+t)^{Np/2}}{c^p} - pt$ Algebraic battle!

$$\boxed{p < 2/N \quad \Rightarrow \quad U^{-p}(t) \overset{t\to\infty}{\sim} -pt}$$



$$\boxed{p > 2/N \quad \Rightarrow \quad U^{-p}(t) \overset{t\to\infty}{\sim} c^{-p}(1+t)^{Np/2}}$$





possible global existence

$$\begin{cases} \partial_t u = \Delta u + v^p, \\ \partial_t v = \Delta v + u^q. \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + t^\alpha v^p, \\ \partial_t v = \Delta v + t^\beta u^q. \end{cases}$$

$$\begin{cases} \partial_t u = \delta \Delta u + v^p, \\ \partial_t v = \Delta v + u^q. \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + u^p v^q, \\ \partial_t v = \Delta v + u^r v^s. \end{cases}$$

$$\left\{ \partial_t u_i = \Delta u_i + u_{i+1}^{p_i}, \quad i \in \frac{\mathbb{Z}}{n\mathbb{Z}}. \right.$$

$$\begin{cases} \partial_t u = \Delta u + u^p + v^q, \\ \partial_t v = \Delta v + u^r + v^s. \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + |x|^\alpha v^p, \\ \partial_t v = \Delta v + |x|^\beta u^q. \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + u^p v^q, \\ \partial_t v = \Delta v + u^r v^s. \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + u^p + v^q, \\ \partial_t v = \Delta v + u^r + v^s. \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + u^p v^q, \\ \partial_t v = \Delta v + u^r v^s. \end{cases}$$

$$\begin{cases} \partial_t u = \Delta u + f_1(t) v^p, \\ \partial_t v = \Delta v + f_2(t) u^q. \end{cases}$$

$$\begin{cases} \partial_t u = J * u - u + v^p, \\ \partial_t v = J * v - v + u^q. \end{cases}$$

1991 1994 1995 1998 2000 2002 2004 2007 2014 2015

Always coupled by the reaction terms

- The aim of this presentation is to consider a Fujita-type problem that is coupled through the diffusion process:

$$\begin{cases} \partial_t u = c\Delta u - \mu u + \nu v + u^{1+p}, & t > 0, \; x \in \mathbb{R}^N, \\ \partial_t v = d\Delta v + \mu u - \nu v + \kappa v^{1+q}, & t > 0, \; x \in \mathbb{R}^N. \end{cases}$$

$\kappa = 0$ or 1

The linear heat-exchanger system

$$(\mathrm{HE})_{\mathrm{diff}} : \begin{cases} \partial_t u = c\Delta u - \mu u + \nu v, & t > 0,\ x \in \mathbb{R}^N, \\ \partial_t v = d\Delta v + \mu u - \nu v, & t > 0,\ x \in \mathbb{R}^N. \end{cases}$$

- First approach : $c = d = \mu = \nu = 1$.

$$\sigma := u + v \qquad \delta := u - v \quad \xrightarrow{\text{uncoupling}} \quad \begin{cases} \partial_t \sigma = \Delta \sigma \\ \partial_t \delta = \Delta \delta - 2\delta \end{cases}$$

$$\Updownarrow$$

$$\xrightarrow{\text{solve}} \quad \begin{cases} \sigma(t) = G(t) * (u_0 + v_0), \\ \delta(t) = e^{-2t}\left[G(t) * (u_0 + v_0)\right] \end{cases}$$

retrieve u and v

$$u = \frac{\sigma + \delta}{2} \qquad v = \frac{\sigma - \delta}{2}$$

$$\begin{cases} u(t) = \frac{1+e^{-2t}}{2}\left[G(t) * u_0\right] + \frac{1-e^{-2t}}{2}\left[G(t) * v_0\right] \\ v(t) = \frac{1-e^{-2t}}{2}\left[G(t) * u_0\right] + \frac{1+e^{-2t}}{2}\left[G(t) * v_0\right] \end{cases} \quad \xrightarrow[\text{exponentially fast}]{t \to \infty} \quad \begin{cases} u_\infty(t) = G(t) * \left(\frac{u_0+v_0}{2}\right) \\ v_\infty(t) = G(t) * \left(\frac{u_0+v_0}{2}\right) \end{cases}$$

Asymptotic behavior of the linear heat-exchanger system

$$(\mathrm{HE})_{\mathrm{diff}} : \begin{cases} \partial_t u = c\Delta u - \mu u + \nu v, & t > 0,\ x \in \mathbb{R}^N, \\ \partial_t v = d\Delta v + \mu u - \nu v, & t > 0,\ x \in \mathbb{R}^N. \end{cases}$$

Theorem (Behavior of the solutions of $(\mathrm{HE})_{\mathrm{diff}}$)

Let (u, v) be the solution to $(\mathrm{HE})_{\mathrm{diff}}$ starting from (u_0, v_0) at $t = 0$. Then for all $t > 1$,

$$\|u(t) - u_\infty(t)\|_{L^\infty(\mathbb{R}^N)} \leq k\, e^{-t\frac{(\sqrt{\mu}+\sqrt{\nu})^2}{2}}, \qquad \|v(t) - v_\infty(t)\|_{L^\infty(\mathbb{R}^N)} \leq k'\, e^{-t\frac{(\sqrt{\mu}+\sqrt{\nu})^2}{2}},$$

where

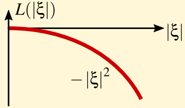
$$\begin{cases} \partial_t u_\infty = \mathcal{L} u_\infty, & t > 0,\ x \in \mathbb{R}^N, \\ u_\infty|_{t=0} = f(u_0, v_0) & x \in \mathbb{R}^N, \end{cases} \qquad \begin{cases} \partial_t v_\infty = \mathcal{L} v_\infty, & t > 0,\ x \in \mathbb{R}^N, \\ v_\infty|_{t=0} = g(u_0, v_0) & x \in \mathbb{R}^N. \end{cases}$$

The diffusive operator $\mathcal{L}$ is characterised through its Fourier transform $\widehat{\mathcal{L}f}(\xi) = L(\xi) \times \widehat{f}(\xi)$ where L is completely known — notice $L(\xi) = -|\xi|^2$ if $\mathcal{L} = \Delta$...

$c = d = \mu = \nu = 1$

$\mathcal{L} = \Delta$

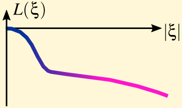
RAS...



$c \ll d \qquad \mu \gg \nu$

$\mathcal{L} \approx \frac{c\nu+d\mu}{\mu+\nu}\Delta$ at low frequencies

$\mathcal{L} \approx c\Delta$ at high frequencies...



f and g are linear combinations of u_0 and v_0 altered by high-pass and low-pass frequency filters.

Decay rate of the linear heat-exchanger

$$(\mathrm{HE})_{\mathrm{diff}} : \begin{cases} \partial_t u = c\Delta u - \mu u + \nu v, & t > 0,\ x \in \mathbb{R}^N, \\ \partial_t v = d\Delta v + \mu u - \nu v, & t > 0,\ x \in \mathbb{R}^N. \end{cases}$$

Corollary (Decay rate)

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \le \frac{k'}{t^{N/2}} \quad \text{and} \quad \|v(t)\|_{L^\infty(\mathbb{R}^N)} \le \frac{k'}{t^{N/2}} \quad \text{for all } t > 0.$$

- Suffices to estimate $u_\infty(t)$ and $v_\infty(t)$ in $L^\infty(\mathbb{R}^N)$

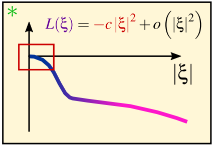
$$\boxed{\|f\|_{L^\infty} \le k\|\widehat{f}\|_{L^1} \ \text{(Hausdorff-Young inequality)}}$$

- Suffices to estimate $\widehat{u}_\infty(t)$ and $\widehat{v}_\infty(t)$ in $L^1(\mathbb{R}^N)$
- Split $\int |\widehat{u}_\infty(t,\cdot)|$ into high and low frequencies:

$$\|\widehat{u}_\infty(t)\|_{L^1} \le \underbrace{\cancel{\|\widehat{u}^{high}_\infty(t)\|_{L^1}}}_{\text{Vanish exponentially fast}} + \|\widehat{u}^{low}_\infty(t)\|_{L^1} \approx \int_{|\xi|<a} |\text{something bounded}| \times e^{tL(\xi)}\, d\xi$$

$$\overset{*}{\lesssim} k \int_{\mathbb{R}^N} e^{-t|\xi|^2}\, d\xi$$

$$= k\left(\frac{\pi}{t}\right)^{N/2}$$



$\kappa = 0$ or 1

$$(\text{HE})_{\text{Fujita}}: \begin{cases} \partial_t u = c\Delta u - \mu u + \nu v + \quad u^{1+p}, & t>0,\ x\in\mathbb{R}^N,\\ \partial_t v = d\Delta v + \mu u - \nu v + \kappa\, v^{1+q}, & t>0,\ x\in\mathbb{R}^N. \end{cases}$$

Theorem (Blow-up vs. global existence for the solutions of $(\text{HE})_{\text{Fujita}}$)

If $$\frac{N}{2} < \begin{cases} \frac{1}{p} & \text{if } \kappa = 0,\\ \max(\frac{1}{p},\frac{1}{q}) & \text{if } \kappa = 1, \end{cases}$$

then **reaction always prevails over diffusion**:
any positive solution to $(\text{HE})_{\text{Fujita}}$ blows-up in finite time.

If $$\frac{N}{2} > \begin{cases} \frac{1}{p} & \text{if } \kappa = 0,\\ \max(\frac{1}{p},\frac{1}{q}) & \text{if } \kappa = 1, \end{cases}$$

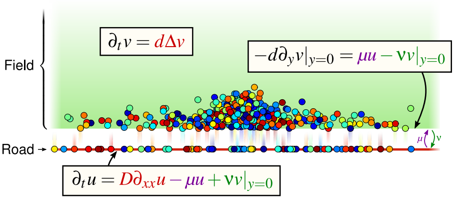
then **diffusion may prevail over reaction**: $(\text{HE})_{\text{Fujita}}$ admits global positive solutions.

- **Possible global existence:** build super-solutions $F(t)\times(U(t,x),V(t,x))$, where (U,V) is the solution to the associated diffusive problem and F a to-be-found function.

- **Systematic blow-up :** pass the solution through an appropriate Gaussian blur and evaluate at place $x=0$. This yields an ODE whom the blowing-up is equivalent to that of (u,v).

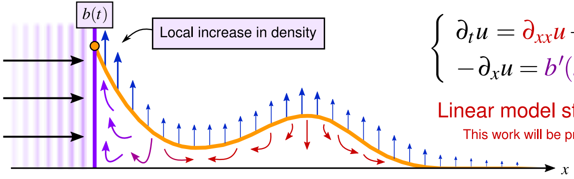
- The influence of a fast diffusion channel (a road) on Fujita exponents

$$\begin{cases} \partial_t v = d\Delta v + v^{1+p}, & t>0, \quad x\in\mathbb{R}^{N-1}, \quad y>0,\\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & t>0, \quad x\in\mathbb{R}^{N-1},\\ \partial_t u = D\Delta_x u - \mu u + \nu v|_{y=0}, & t>0, \quad x\in\mathbb{R}^{N-1}. \end{cases}$$



L^∞ estimates established for the linear model in [*Alfaro, Ducasse, Tréton – JDE 2023*]

- The influence of an adiabatic inward-shifting boundary (a piston) on Fujita exponents



$$\begin{cases} \partial_t u = \partial_{xx} u + u^{1+p} & t>0 \quad x>b(t)\\ -\partial_x u = b'(t)u & t>0 \quad x=b(t) \end{cases}$$

Linear model studied in [*Tréton, Zhang – arXiv 2025*]

This work will be presented tomorrow (18:30 → 19:00, salle Saint-Julien)

Thanks for your attention!

The original paper of Fujita

Fujita
On the blowing up of solutions of the Cauchy problem for $\partial_t u = \Delta u + u^{1+\alpha}$
Journal of the Faculty of Science, University of Tokyo (1966)

Extinction with Allee effect

Aronson and Weinberger
Multidimensional nonlinear diffusion arising in population genetics
Advances in Mathematics (1978)

A well known book on the Fujita blow-up phenomena

Quittner and Souplet
Superlinear Parabolic Problems
Springer International Publishing (2019)

Fujita on the Heat exchanger

Tréton
Blow-up vs. global existence for a Fujita-type Heat exchanger system
SIAM Journal of Mathematical Analysis (2023)

The field-road model

Alfaro, Ducasse and Tréton
The field-road diffusion model: fundamental solution and asymptotic behavior
Journal of Differential Equations (2023)

The piston model

Tréton and Zhang
A piston to counteract diffusion: the influence of an inward-shifting boundary on the heat equation in half-space
arXiv:2505.03304 (2025)

This work will be presented tomorrow (18:30 $\rightarrow$ 19:00, salle Saint-Julien)

Fujita: possible global existence

$$(\star)\ \ \partial_t u = \Delta u + u^{1+p}, \qquad t > 0,\ \ x \in \mathbb{R}^N$$

Theorem (H.Fujita) (1966)

If $\frac{1}{p} < \frac{N}{2}$, then **diffusion may prevail over reaction** : $(\star)$ admits global positive solutions.

- Build a global super solution: $\overline{u}(t,x) = f(t) \times \mathrm{u}(t,x)$

  to be found

  solution to the Heat equation

- We ask then $\partial_t \overline{u} \geq \Delta \overline{u} + \overline{u}^{1+p}$

- A sufficient condition to get that is $f'(t) = \dfrac{c(\|u_0\|_{L^1} + \|u_0\|_{L^\infty})}{(1+t)^{Np/2}} \times f^{1+p}(t)$

- By solving we find f which is bounded if

  - $p > 2/N$

  and

  - $\|u_0\|_{L^1} + \|u_0\|_{L^\infty}$ sufficiently small...

$$(\star)\quad \partial_t u = \Delta u + u^{1+p}, \qquad t > 0, \quad x \in \mathbb{R}^N$$

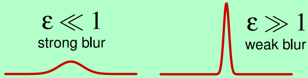
Theorem (H.Fujita) (1966)

If $\frac{1}{p} > \frac{N}{2}$, then **reaction always prevails over diffusion**: any positive solution to $(\star)$ blows-up in finite time.

- Introduce for $\varepsilon > 0$, $\Phi_\varepsilon(x) := \left(\frac{\varepsilon}{\pi}\right)^{N/2} \times e^{-\varepsilon|x|^2}$, so that $\|\Phi_\varepsilon\|_{L^1} = 1$ for any ε.

For $\lambda > 0$, $(\Delta - \lambda)\Phi_\varepsilon = (4\varepsilon^2|x|^2 - 2\varepsilon N + \lambda)\Phi_\varepsilon \geq (-2\varepsilon N + \lambda)\Phi_\varepsilon$

so that, for $\lambda := 2N\varepsilon$, $\quad \Delta\Phi_\varepsilon \geq -\lambda\Phi_\varepsilon \quad (\star)$



- Call U_ε the solution u blurred by convolution with Φ_ε evaluated at $x = 0$:

$$U_\varepsilon(t) := [\Phi_\varepsilon * u(t,\cdot)]\,(x=0) = \int_{\mathbb{R}^N} \Phi_\varepsilon(z)u(t,z)dz$$

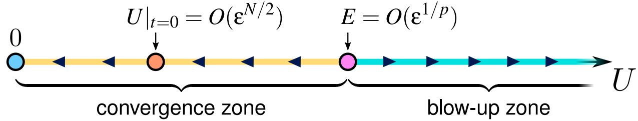
then

$$\underbrace{U_\varepsilon(t)}_{\text{remains to make this blow-up!}} \leq \|u(t)\|_{L^\infty}$$

- $$U_\varepsilon' = \Phi_\varepsilon * (\Delta u + u^{1+p}) \overset{\text{IBP}}{=} \Delta\Phi_\varepsilon * u + \Phi_\varepsilon * (u^{1+p}) \underset{+(\star)}{\overset{\text{Jensen}}{\geq}} -\lambda\Phi_\varepsilon * u + (\Phi_\varepsilon * u)^{1+p} = -\lambda U_\varepsilon + U_\varepsilon^{1+p}$$

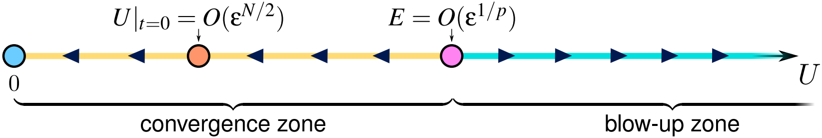
Hence, $U_\varepsilon > U$, where

$$\begin{cases} U' = -\lambda U + U^{1+p}, \\ U|_{t=0} = U_0 = O(\varepsilon^{N/2}) \end{cases}$$



Scalar case

- $\Phi_\varepsilon(x) := \left(\frac{\varepsilon}{\pi}\right) e^{-\varepsilon|x|^2}$
- $U(t) := [\Phi_\varepsilon * u(t,\bullet)](0)$
- U blows-up $\Rightarrow$ u blows-up
- $U'(t) \geq -2N\varepsilon U + U^{1+p}$



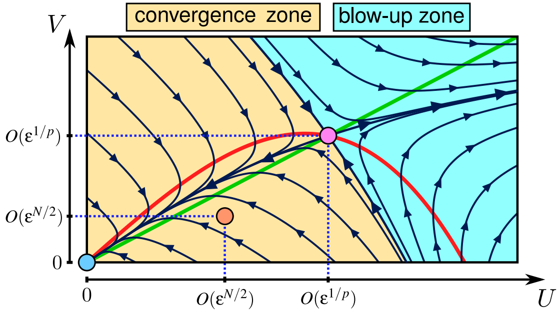
- For ε small enough, $U|_{t=0} > E$ when $\frac{1}{p} > \frac{2}{N}$

Heat exchanger

$$U(t) := [\Phi_\varepsilon * u(t,\bullet)](0)$$
$$V(t) := [\Phi_\varepsilon * v(t,\bullet)](0)$$

$$\begin{cases} U' \geq -(\mu + 2cN\varepsilon)U + \nu V + U^{1+p} \\ V' \geq \mu u - (\nu + 2dN\varepsilon)V \end{cases}$$

- For ε small enough, $(U,V)|_{t=0} =$ ● is in blow-up zone when $\frac{1}{p} > \frac{2}{N}$



$$(\mathrm{HE})_{\mathrm{diff}}: \begin{cases} \partial_t u = c\Delta u - \mu u + \nu v, & t>0,\ x\in\mathbb{R}^N, \\ \partial_t v = d\Delta v + \mu u - \nu v, & t>0,\ x\in\mathbb{R}^N. \end{cases}$$

$$\partial_t \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \underbrace{\begin{pmatrix} -c|\xi|^2-\mu & \nu \\ \mu & -d|\xi|^2-\nu \end{pmatrix}}_{=:A(\xi)} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$$

λ_+ and λ_- the two eigenvalues of $A(\xi)$

- Explicit solution in Fourier

$$\begin{pmatrix} \hat{u}(t,\xi) \\ \hat{v}(t,\xi) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \left(1-\frac{r}{\sqrt{s}}\right)e^{t\lambda_+} + \left(1+\frac{r}{\sqrt{s}}\right)e^{t\lambda_-} & \frac{\nu}{\sqrt{s}}\,e^{t\lambda} - \frac{\nu}{\sqrt{s}}\,e^{t\lambda_-} \\ \frac{\mu}{\sqrt{s}}\,e^{t\lambda_+} - \frac{\mu}{\sqrt{s}}\,e^{t\lambda_-} & \left(1+\frac{r}{\sqrt{s}}\right)e^{t\lambda_+} + \left(1-\frac{r}{\sqrt{s}}\right)e^{t\lambda_-} \end{pmatrix} \begin{pmatrix} \hat{u}_0(\xi) \\ \hat{v}_0(\xi) \end{pmatrix}$$

$$= \begin{pmatrix} \hat{u}_e(t,x) \\ \hat{v}_e(t,x) \end{pmatrix} + \begin{pmatrix} \hat{u}_\infty(t,x) \\ \hat{v}_\infty(t,x) \end{pmatrix}$$

(evanescent) + (persistent)

- Hausdorff–Young inequality $|u(t,x)| = \left|\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\hat{u}(t,\xi)e^{i\xi\cdot x}d\xi\right| \leq \frac{1}{(2\pi)^N}\|\hat{u}(t,\bullet)\|_{L^1(\mathbb{R})}$
- For u (the same for v):

$$\|u_e(t,\bullet)\|_{L^\infty(\mathbb{R}^N)} \leq k\|\hat{u}_e(t,\bullet)\|_{L^1(\mathbb{R}^N)} \leq \tilde{k}e^{-\lambda t}$$ exponential vanishing

$$\|u_\infty(t,\bullet)\|_{L^\infty(\mathbb{R}^N)} \leq \ell\|\hat{u}_\infty(t,\bullet)\|_{L^1(\mathbb{R}^N)} \leq \tilde{\ell}/t^{N/2}$$ algebraical vanishing