A piston to counteract diffusion: The influence of an inward-shifting boundary on the heat equation in half-space

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Climate change, among other environmental factors, has an increasing impact on the distribution of biological populations. To better understand how these populations respond to dynamic external pressures, we propose a new diffusion model in the moving half-line $\Omega_t := \{z > b(t)\}$, where the boundary position $b(t)$ is a given, smooth and increasing function of time. By imposing a suitable (Robin-type) boundary condition at $z = b(t)$, we prevent individuals from leaving the domain, so that the shifting boundary acts as an impermeable wall —a "piston"— that sweeps the individuals it encounters. This framework leads to an intricate interplay between the diffusion mechanism (which tends to spread the population) and the accumulation of individuals against the boundary.
As it is natural to consider algebraic speeds for the boundary, we focus here on cases where $b(t) \sim ct^\beta$ as $t \to \infty$. Our main contribution is a complete characterization of the long-time distribution of the population for any $\beta \in [0, 1]$. Notably, the asymptotic solution switches from a Gaussian shape to an exponential shape at $\beta = 1/2$, and converge to a nontrivial steady state in the special case $\beta = 1$. In this latter scenario, the dispersal effect of the Laplacian is perfectly balanced by the accumulation of individuals at the moving boundary. These new effects observed in the context of this original diffusion model open promising perspectives for incorporating density-dependent growth terms, which could account for more complex population dynamics, such as resource limitations, overcrowding dynamics, or on the contrary, Allee effect rescues. We intend to explore such directions in future work.

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