A weighted finite volume scheme for growth-fragmentation models

Christèle ETCHEGARAY, Centre Inria de l'université de Bordeaux - Talence
Julien GRANET, Institut de Mathématiques de Bordeaux - Talence
Clair POIGNARD, Centre Inria de l'université de Rennes - Rennes

We are interested in the numerical discretization of the growth-fragmentation equation. Let $u(t,x)$ be the density of individuals at time $t > 0$ and of mass $x \in [0, x_M]$, the equation then writes

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}\left(\tau(x)u(t,x)\right) = \int_x^{x_M} \beta(y)\kappa(x,y)u(t,y)\mathrm{d}y - \beta(x)u(t,x),$$

where $\tau(x) \geq 0$ is the growth rate of an entity of mass x, $\beta(x) \geq 0$ is the frequency of a fragmentation event for an individual of mass x and $\kappa(x,y) \geq 0$ is the kernel for fragmentation of an entity of mass y into an entity of size $x < y$. This type of model typically describes the time evolution of a mass-structured population from the growth and division of its individuals. Integral properties of said population provide macroscopic quantities that may help calibrating the model on experimental data. More precisely, we are interested in the 0th and 1st moments of the solution u given by

$$\mathcal{M}^{[n]}(t) = \int_0^{x_M} x^n u(t,x)\mathrm{d}x, \qquad n \in \{0, 1\},$$

corresponding respectively to the total population density and the total population mass. In practice, simultaneously recovering u and its associated moments $\mathcal{M}$ in numerical simulations can prove to be tricky, as we have to compute them from a truncation of the solution u. This will, in general, result in the loss of consistency of the integral quantities. In [1], Kumar and al. proposed a weighted numerical scheme that successfully approximates u and $\mathcal{M}^{[1]}$ in the case of pure fragmentation ($\tau = 0$).
In this talk, we will introduce a new finite volume scheme corrected with weights. Our method expands the results given in [1], in the sense that it applies to the general growth-fragmentation equation, and allows to retrieve the solution u and both moments $\mathcal{M}^{[0]}$ and $\mathcal{M}^{[1]}$. The expression of such weights will be given for a simple order 1 scheme, and adaptation to higher orders will be discussed.

[1] J. Kumar, J. Saha, E. Tsotsas. *Development and convergence analysis of a finite volume scheme for solving breakage equation.* SIAM Journal on Numerical Analysis, **53**, 1672–1689, 2015. doi : 10.1137/140980247.

Contact : `julien.granet@math.u-bordeaux.fr`