On an equation for the evolution of an urban area

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A variational principal for the organization of urban areas

Propose an equation for the evolution of the population and working sites of an urban area via variational principles

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Population distributed as $\mathscr{P}(\Omega) \ni \varrho \to \min \int_\Omega F(\varrho(x))\mathrm{d}x + W_2^2\left(\varrho, \sum_{i=1}^{N} a_i \delta_{x_i}\right)$

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Working sites given by $(\mathbf{x}, \mathbf{a}) \to \min W_2^2\left(\varrho, \sum_{i=1}^N a_i \delta_{x_i}\right) + \sum_{i=1}^N g(a_i)$

where $\mathbf{x} = (x_i)_{i=1}^N \subset \Omega^{\otimes N}$, $\mathbf{a} = (a_i)_{i=1}^N \in \Delta_{N-1}$

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A variational principal for the organization of urban areas

Energy proposed by Buttazzo and Santambrogio for the configuration of urban areas[1]

$$\mathscr{E}(\varrho, \mathbf{x}, \mathbf{a}) \overset{\text{def.}}{=} \int_\Omega F(\varrho(x))\mathrm{d}x + \sum_{i=1}^N g(a_i) + W_2^2\left(\varrho, \sum_{i=1}^N a_i \delta_{x_i}\right)$$

$$\mathscr{F}(\varrho) \overset{\text{def.}}{=} \int_\Omega F(\varrho(x))\mathrm{d}x \to \text{ forces uniform integrability}$$

[1] Giuseppe Buttazzo and Filippo Santambrogio. "A model for the optimal planning of an urban area". In: *SIAM journal on mathematical analysis* (2005).

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$$\mathscr{G}(\mu_{\mathbf{x},\mathbf{a}}) \overset{\text{def.}}{=} \sum_{i=1}^N g(a_i) \rightarrow \text{ encourages economy of scale as } \lim_{t \to 0+} \frac{g(t)}{t} = +\infty$$

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$$W_2^2(\mu, \nu) \stackrel{\text{def.}}{=} \min_{T_\sharp \mu = \nu} \frac{1}{2} \int_\Omega |x - T(x)|^2 \mathrm{d}\mu \rightarrow \text{ penalizes total transportation cost}$$

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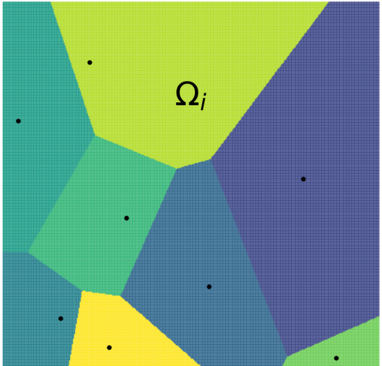
A variational principal for the organization of urban areas

In the semi-discrete case we have the formation of Laguerre tesselations

$$W_2^2\left(\varrho, \sum_{i=1}^N a_i \delta_{x_i}\right) = \max_{\psi \in \mathbb{R}^N} \int_\Omega \min_{i=1,\dots,N} \left[\frac{1}{2}|x - x_i|^2 - \psi_i\right] \mathrm{d}\varrho + \sum_{i=1}^N a_i \psi_i$$



$$\Omega_i = \mathrm{Lag}_i(\mathbf{x}, \psi) \stackrel{\text{def.}}{=} \left\{ x \in \Omega : \frac{1}{2}|x - x_i|^2 - \psi_i \leq \frac{1}{2}|x - x_j|^2 - \psi_j \text{ for all } j \right\}$$

A dynamical model

This is a static formulation

$$\min_{\varrho,\mathbf{x},\mathbf{a}} \mathscr{E}(\varrho, \mathbf{x}, \mathbf{a})$$

Consider the gradient flow of the energy $\mathscr{E}$ ...

A dynamical model

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$$\begin{aligned}
&\partial_t \varrho_t = \operatorname{div}\left(\varrho_t\left(\nabla F'(\varrho_t) + \sum_{i=1}^N (x - x_i(t))1_{\Omega_i(t)}\right)\right)\\
&0 = n_\Omega \cdot \left(\nabla F'(\varrho_t) + \sum_{i=1}^N (x - x_i(t))1_{\Omega_i(t)}\right)\\
&\dot{x}_i \in -a_i x_i + \int_{\Omega_i(t)} x \mathrm{d}\varrho_t + N_\Omega(x_i),\\
&\dot{a}_i = -g'(a_i) - \psi_i(t), \text{ over } a_i > 0,\\
&\ 0 = \sum_{i:a_i>0} \psi_i(t) + \sum_{i:a_i>0} g'(a_i), \quad \Omega_i(t) = \mathrm{Lag}_i(\psi_t, \mathbf{x}_t),\\
&\psi_t = (\psi_i(t))_{i=1}^N \text{ is a potential for } W_2^2(\varrho_t, \mu_t)
\end{aligned} \tag{GradFlow}$$

A dynamical model

Consider the gradient flow of the energy $\mathscr{E}$ ...

$$
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&0 = \sum_{i:a_i>0} \psi_i(t) + \sum_{i:a_i>0} g'(a_i), \quad \Omega_i(t) = \mathrm{Lag}_i(\psi_t, \mathbf{x}_t),\\
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\end{aligned}
\tag{GradFlow}
$$

A dynamical model

Consider the gradient flow of the energy $\mathscr{E}$… or more simply

$$\begin{aligned}
&\partial_t \varrho_t = \operatorname{div}\left(\varrho_t\left(\nabla F'(\varrho_t) + \sum_{i=1}^N (x - x_i(t))1_{\Omega_i(t)}\right)\right)\\
&\dot{x}_i \in -a_i x_i + \int_{\Omega_i(t)} x \mathrm{d}\varrho_t,\\
&\dot{a}_i = -g'(a_i) - \psi_i(t), \text{ over } a_i > 0,
\end{aligned} \tag{GradFlow}$$

Definition

We say $t \mapsto (\varrho_t, \mathbf{x}_t, \mathbf{a}_t) \in \mathscr{C}^0(\mathscr{P}_2 \times \Omega^{\otimes N} \times \Delta_{N-1})$ is a weak solution to (GradFlow) s.t. $F'(\varrho) \in L^1([0,T] \times W^{1,1}(\Omega))$, satisfying the boundary conditions and

- $$0 = \int_0^T \int_\Omega \left(-\partial_t \phi + \nabla \phi \cdot \left[\nabla F'(\varrho_t) + \sum_{i=1}^N (x - x_{t,i}^\tau) 1_{\Omega_i(t)} \right] \right) \mathrm{d}\varrho_t \mathrm{d}t$$
  for all $\phi \in \mathscr{C}_c^\infty((0,T) \times \Omega)$
- $$-\int_0^T f'(t) \cdot x_{t,i} \mathrm{d}t = -\int_0^T f(t) \cdot \left(a_{t,i} \cdot x_{t,i} - \int_{\Omega_{t,i}} x \mathrm{d}\varrho_t \right) \mathrm{d}t$$
  for all $f \in \mathscr{C}_c^\infty((0,T), \mathbb{R}^d)$ and $i = 1, \ldots, N$
- $$-\int_0^T h'(t) a_i(t) \mathrm{d}t = -\int_0^T h(t) \cdot (g'(a_i(t)) + \psi_i(t)) \, \mathrm{d}t$$
  for all $h \in \mathscr{C}_c^\infty((0,T))$ and $i = 1, \ldots, N$

Existence of Solutions: the JKO scheme

Via the JKO scheme[1]

$$\varrho^\tau_{k+1} \in \operatorname{argmin} \mathscr{F}(\varrho) + \frac{1}{2\tau} W_2^2(\varrho^\tau_k, \varrho), \quad \varrho^\tau_t \overset{\text{def.}}{=} \varrho^\tau_{k+1} \text{ if } t \in (k\tau, (k+1)\tau]$$

[1] Jordan, Kinderlehrer, and Otto, "The variational formulation of the Fokker–Planck equation".

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Then we know that ϱ^{τ} converges to a weak solution of

$$\partial_t \varrho_t = \operatorname{div}\left(\varrho_t \nabla \frac{\delta \mathscr{F}}{\delta \varrho}(\varrho_t)\right), \quad n_{\Omega} \cdot \nabla \frac{\delta \mathscr{F}}{\delta \varrho}(\varrho_t) = 0$$

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$$\varrho_{k+1}^\tau \in \operatorname{argmin} \mathscr{F}(\varrho) + \frac{1}{2\tau} W_2^2(\varrho_k^\tau, \varrho), \quad \varrho_t^\tau \stackrel{\text{def.}}{=} \varrho_{k+1}^\tau \text{ if } t \in (k\tau, (k+1)\tau]$$

Then we know that ϱ^τ converges to a weak solution of

$$\mathscr{F}(\varrho) = \int_\Omega \varrho \log \varrho \mathrm{d}x, \quad \partial_t \varrho_t = \Delta \varrho \text{ Heat equation}$$

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Existence of Solutions: the JKO scheme

Via the JKO scheme[1]

$$\varrho_{k+1}^{\tau} \in \operatorname{argmin} \mathscr{F}(\varrho) + \frac{1}{2\tau} W_2^2(\varrho_k^{\tau}, \varrho), \quad \varrho_t^{\tau} \overset{\text{def.}}{=} \varrho_{k+1}^{\tau} \text{ if } t \in (k\tau, (k+1)\tau]$$

Then we know that ϱ^{τ} converges to a weak solution of

$$\mathscr{F}(\varrho) = \frac{1}{m-1} \int_{\Omega} \varrho^m \mathrm{d}x, \quad \partial_t \varrho_t = \Delta \varrho^m \text{ Porous medium equation}$$

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Existence of Solutions: the JKO scheme

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Strategy used to show existence of solutions to other non-linear PDEs

- in crowd motion[2]
- for the total-variation flow[3]

[1] Jordan, Kinderlehrer, and Otto, "The variational formulation of the Fokker–Planck equation".

[2] Maury et al., "Handling congestion in crowd motion modeling".

[3] Carlier and Poon, "On the total variation Wasserstein gradient flow and the TV-JKO scheme".

Convergence of the JKO scheme

$$\left(\varrho_{k+1}^{\tau}, \mathbf{x}_{k+1}^{\tau}, \mathbf{a}_{k+1}^{\tau}\right)_{k\in\mathbb{N}} \in \operatorname{argmin} \mathscr{F}(\varrho) + \mathscr{G}(\mu_{\mathbf{x},\mathbf{a}}) + W_2^2(\varrho, \mu_{\mathbf{x},\mathbf{a}}) + \frac{1}{2\tau}\left(W_2^2(\varrho_k^{\tau}, \varrho) + \|\mathbf{x}_k^{\tau} - \mathbf{x}\|^2 + \|\mathbf{a}_k^{\tau} - \mathbf{a}\|^2\right).$$

Theorem

Let Ω be a bounded convex subset of $\mathbb{R}^d$. If $\mathscr{E}(\varrho_0, \mathbf{x}_0, \mathbf{a}_0) < +\infty$ and either

- *Ω has a $\mathscr{C}^1$ boundary,*
- *Ω has a Lipschitz boundary and the initial conditions satisfy $x_i(0) \in \operatorname{int}\Omega$, $a_i(0) > 0$ for all $i = 1, \dots, N$*

the interpolations of the JKO scheme $(\varrho^{\tau}, \mathbf{x}^{\tau}, \mathbf{a}^{\tau})_{\tau>0}$ admits subsequences converging in $\mathscr{C}^{0,1/2}([0,1], \mathscr{P}(\Omega) \times \Omega^{\otimes N} \times \Delta_{N-1})$ to a weak solution of (GradFlow).

Difficulties:

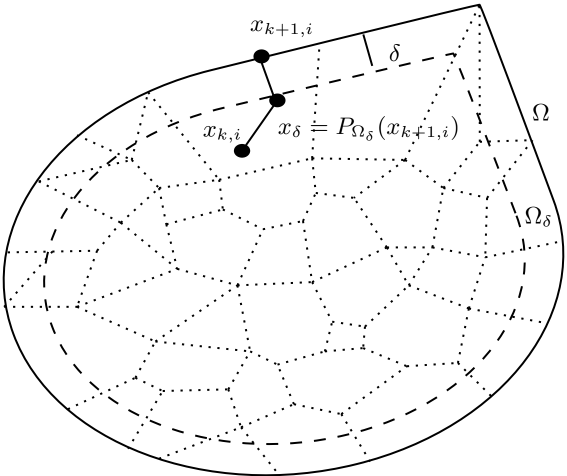
1. How to deal with the boundary effects when $x_i \in \partial\Omega$, specially for $\partial\Omega$ Lipschitz? Can we expect a better behavior for the limit PDE?
2. How to characterize the derivative of a_i as

$$\dot{a}_i = -g'(a_i) - \psi_i(t)$$

when $a_i(t) \to 0$ since $g'(a_i(t)) \to +\infty$.

Solutions:

1. Uniform integrability of ϱ^τ_{k+1}: if $a^\tau_{i,k+1} > 0$ and $x^\tau_{i,k} \in \operatorname{int}\Omega$, then $x^\tau_{i,k+1} \in \operatorname{int}\Omega$



Solutions:

2 If $t \mapsto a_i(t)$ is a limit trajectory of the JKO, if holds that

$$a_i(t) = 0, \text{ then } a_i(s) = 0 \text{ for all } s > t$$

Hence it suffices to determine the dynamics of $a_i(\cdot)$ over $(0, t_i)$

$$t_i \stackrel{\text{def.}}{=} \inf \{t \geq 0 : a_i(t) = 0\}$$

Convergence of the JKO scheme: Strong $L_t^2 H_x^1$

Santambrogio et.al. proved recently strong $L_t^2 H_x^2$ convergence of the JKO for Fokker Planck with smooth potentials.[4]

Theorem

- *When* $F(\rho) = \rho \log \rho$, *the family* $(\varrho^\tau)_{\tau>0}$, *up to subsequences, converges strongly in* $L_t^2 H_x^1$ *to* ϱ, *where* $(\varrho, \mathbf{x}, \mathbf{a})$ *is a solution to (GradFlow).*
- *When* $F(\rho) = \frac{1}{m-1}\rho^m$, *the family of pressures* $((\varrho^\tau)^m)_{\tau>0}$, *up to subsequences, converges strongly in* $L_t^2 H_x^1$ *to* ϱ^m, *where* $(\varrho, \mathbf{x}, \mathbf{a})$ *is a solution to (GradFlow).*

[4] Santambrogio and Toshpulatov, "Strong $L^2 H^2$ Convergence of the JKO Scheme for the Fokker–Planck Equation".

Convergence of the JKO scheme: Strong $L^2_t H^1_x$

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- *When $F(\rho) = \frac{1}{m-1}\rho^m$, the family of pressures $((\varrho^\tau)^m)_{\tau>0}$, up to subsequences, converges strongly in $L^2_t H^1_x$ to ϱ^m, where $(\varrho, \mathbf{x}, \mathbf{a})$ is a solution to (GradFlow).*

In particular, the JKO solution $\varrho \in L^2([0,T]; H^1(\Omega))$!

[4] Santambrogio and Toshpulatov, "Strong $L^2 H^2$ Convergence of the JKO Scheme for the Fokker–Planck Equation".

Some qualitative properties

Theorem

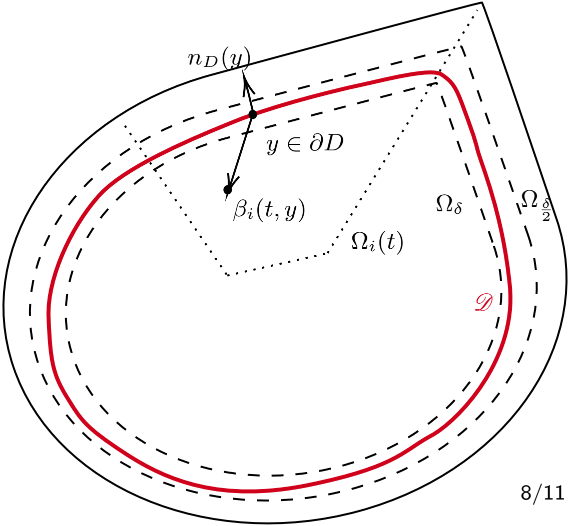
- *Assume $\partial\Omega$ is $\mathscr{C}^1$. If $a_i(0) > 0$ and $x_i(0) \in \partial\Omega$, then $x_i(t) \in \operatorname{int}\Omega$ for $t > 0$ small enough.*

  x_i is immediately pushed towards the interior.
- *Assume $\partial\Omega$ is $\mathscr{C}^1$ or that $x_i(0) \in \operatorname{int}\Omega$. If $x_i(t) \in \partial\Omega$, then $a_i(t) = 0$.*

  Particles can only be destroyed, but not created or split.

Proof by contradiction and construction of an invariant region for the dynamics

$$\dot{z}_i(t) = -a_i(t) z_i + \underbrace{\int_{\Omega_i(t,z_i)} x \,\mathrm{d}\varrho_t(x)}_{=a_i(t)\beta_i(t)}$$



Asymptotic Behavior

Previous remarks justify the long time study of a simpler equation, which can be seen as a dynamic quantization equation

$$\begin{cases} \partial_t \varrho_t = \Delta \varrho_t + \operatorname{div}\left(\varrho_t \sum_{i=1}^{N} (x - x_i) 1_{\Omega_i}\right) \\ \dot{x}_i = -\frac{1}{N} x_i + \int_{\Omega_i} x \mathrm{d}\varrho_t \end{cases}$$

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The stationary measures are given by the following equilibrium criteria

$$\mathrm{d}\varrho_\infty(x) = \frac{1}{Z} e^{-\Phi_\infty} \mathrm{d}x, \text{ where } \Phi_\infty(x) = \sum_{i=1}^{N} \left(\frac{1}{2} |x - x_{i,\infty}|^2 - \psi_{i,\infty} \right) 1_{\Omega_i}$$

$$x_{i,\infty} = N \int_{\mathrm{Lag}_i(\mathbf{x}_\infty, \psi_\infty)} x \mathrm{d}\varrho_\infty(x)$$

Asymptotic Behavior: some heuristics

$$\int_\Omega \left\|\nabla \log \varrho_t + \nabla \Phi_t\right\|^2 \mathrm{d}\varrho_t \xrightarrow[t\to\infty]{} 0$$ variant of Bakry-Émery method [5]

$$\left\| x_i(t) - N \int_{\mathrm{Lag}_i(\mathbf{x}_t,\psi_t)} x \mathrm{d}\varrho_t(x) \right\| \xrightarrow[t\to\infty]{} 0$$ continuous dynamics of Loyd's algorithm [6]

[5] Dominique Bakry and Michel Émery. "Diffusions hypercontractives". In: *Séminaire de Probabilités XIX 1983/84: Proceedings*. 2006.

[6] Quentin Merigot, Filippo Santambrogio, and Clement Sarrazin. "Non-asymptotic convergence bounds for Wasserstein approximation using point clouds". In: *Advances in Neural Information Processing Systems* (2021).

Thanks for your attention !