Periodic homogenization and harmonic measures

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Consider the Dirichlet problem

$$\begin{cases} -\nabla \cdot A\nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where A is a uniformly elliptic matrix field and $\Omega \subset \mathbb{R}^d$ is a bounded domain. The classical approach based on the trace operator requires $f \in H^{\frac{1}{2}}(\partial\Omega)$, and cannot be easily extended to rougher boundary data —for instance, $f \in L^2(\partial\Omega)$. The solvability of (1) with L^p boundary data turns out to be subtle and depends intricately on both the boundary behavior of A and the geometric features of Ω.
Since the seminal work of Kenig and Pipher [2], the so-called Dahlberg-Kenig-Pipher (DKP) condition on oscillations of the coefficient matrix has become a standard threshold in the study of L^p-solvability of the Dirichelet problem on bounded domains. It has been proved sufficient for the L^p-solvability in the domains with increasingly complex geometry, and known counterexamples show that in a certain sense it is necessary as well. More precisely, the DKP condition requires that, with $x \in \partial\Omega$ and $\delta(X) := \mathrm{dist}(X, \partial\Omega)$,

$$\int_{B_r(x)\cap\Omega} \delta(X) \sup_{B_{\delta(X)/2}(X)} |\nabla A|^2 dX \leq Cr^{d-1}. \tag{2}$$

One can see that, by

$$\sup_{Y,Y' \in B_{\delta(X)/2}(X)} |A(Y) - A(Y')| \lesssim \delta(X) \sup_{B_{\delta(X)/2}(X)} |\nabla A|,$$

the DKP condition (2) implies that A oscillates mildly near the boundary and, therefore, the associated elliptic operator behaves like the Laplacian.
In the presentation, by putting the geometric complications aside, we present the class of operators introduced in [1], for which the L^p-solvability of (1) is established. These operators feature the coefficients that violate the DKP condition which controls local oscillations of A, and, on the contrary, oscillate so quickly that the homogenization ensures that, in the resolvent sense, the operators still behave like the Laplacian near the boundary.

[1] G. David, A. Gloria, S. Mayboroda, S. Qi. *Periodic homogenization and harmonic measures*, 2025. ArXiv:2504.17396.

[2] C. E. Kenig, J. Pipher. *The Dirichlet problem for elliptic equations with drift terms.* Publ. Mat., **45(1)**, 199–217, 2001. doi:10.5565/PUBLMAT_45101_09.

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