Une caractérisation "simple" de *GBD*

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SMAI 2025

Topic: the space *GBD*

Was Introduced by G. Dal Maso to tackle the minimization of *Griffith's energy*:

$$\min_{(u,K)} \int_{\Omega\setminus K} \mathbb{C}e(u) : e(u)dx + \mathcal{H}^{d-1}(K) =: \mathcal{E}(u,K)$$

subject to $u = U^0$ on $\partial^D\Omega \setminus K$, where $K \subset \Omega$ is a crack set (of co-dimension 1), $u : \Omega \to \mathbb{R}^d$ is an infinitesimal displacement, and $e(u) = (Du + Du^T)/2$ is the symmetrized gradient. ($\mathbb{C}$ is the "Hooke's law", typically $\mathbb{C}e(u) = \lambda(\mathrm{Tr}e(u))I + 2\mu e(u)$ for an isotropic linear elastic material).

[Problem introduced in the late 90's by Francfort and Marigo to model crack growth in linearized elasticity.]

Difficulty: Energy space for $\mathcal{E}(u,K)$?

Minimizing sequences

If (u_n, K_n) is a minimizing sequence for $\mathcal{E}$, $K_n \to K$ in the Hausdorff sense (up to a subsequence), $u_n \to u$ in $H^1_{loc}(\Omega \setminus K)$ and (easy)

$$\int_{\Omega\setminus K} \mathbb{C}e(u) : e(u)dx \leq \liminf_n \int_{\Omega\setminus K_n} \mathbb{C}e(u_n) : e(u_n)dx.$$

Yet, with this approach, very hard (and not true in general) to show:

$$\mathcal{H}^{d-1}(K) \leq \liminf_n \mathcal{H}^{d-1}(K_n).$$

- True if K_n, K are assumed connected, in 2D;
- For the "Mumford-Shah functional" which is the scalar variant: 2D by Dal Maso-Morel-Solimini (early 90's) [first bound the number of components then send it to $+\infty$];
- Still Mumford-Shah, any dimension: modify K_n by removing low density pieces: Maddalena-Solimini (early 2000's).

Weak formulation

Alternatively, one introduces a “weak formulation”:

$$\mathcal{E}(u) = \int_\Omega \mathbb{C}e(u) : e(u)dx + \mathcal{H}^{d-1}(J_u)$$

where J_u is the *jump set* of u, that is the set of points

$$J_u = \left\{ x \in \Omega : u(x+ry) \xrightarrow{L^1(B_1)} u^+(x)\chi_{\{y\cdot\nu_u(x)\geq 0\}} + u^-(x)\chi_{\{y\cdot\nu_u(x)<0\}} \right\}$$

which is a rectifiable $(d-1)$-dimensional set (Del Nin, 2021)
Issue: for which functions can we define J_u (and $e(u)$ out of J_u)?

BV, SBV, $GSBV$, BD, SBD...

In the scalar setting, the relevant space is BV, the space of functions with bounded variations, whose gradient Du is a bounded Radon measure. One can show that $\mathcal{H}^{d-1}$-a.e. point is either a jump point (J_u) or a Lebesgue point, and Du is decomposed as

$$Du = \nabla u(x)dx + Cu + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{d-1}|_{J_u}$$

with Cu the "Cantor part" which is "in between" dimensions d and $d-1$. Then, "SBV" are the functions for which $Cu = 0$, and the Mumford-Shah energy is coercive and lsc. in SBV (*Ambrosio's compactness theorem*) provided the minimizing sequence is *uniformly bounded*.

Reason: $\mathcal{H}^{d-1}(J_u)$ does not control, in general, $\int_{J_u} |u^+ - u^-| d\mathcal{H}^{d-1}$ which is the mass of the jump part of the differential $\rightarrow$ "GBV", "$GSBV$" defined by truncation.

BV, SBV, $GSBV$, BD, SBD...

In the vectorial linearized elasticity setting, BV has to be replaced with "BD" the set of displacements $u : \Omega \to \mathbb{R}^d$ such that the *symmetrized* gradient $Eu = (Du + Du^T)/2$ is a Radon measure. Then, one has:

$$Eu = e(u)(x)dx + Cu + (u^+ - u^-) \odot \nu_u \mathcal{H}^{d-1}|_{J_u}$$

where $a \odot b = (a \otimes b + b \otimes a)/2$ is the symmetrized tensor product of two vectors.

Again: "SBD" if $Cu = 0$, is enough to minimize the weak Griffith energy with an additional constraint $\|u\|_\infty \leq C$.

But in general? No a priori L^∞ bound. No easy strategy as in the scalar case (truncate, show maximum principle...)

$G(S)BD$?

No “working” definition until Dal Maso’s suggestion around 2010. Idea of Dal Maso: use 1D “slices”.

1D Slicing of BV or BD functions

Indeed, $u \in BV$ if and only if for all $\xi \in \mathbb{S}^{d-1}$ and almost all $z \in \xi^\perp$,

$$u_{\xi,z} : s \mapsto u(z + s\xi) \text{ is } BV, \quad \text{and}$$
$$|Du| \approx \max_\xi \int_{\xi^\perp} |Du_{\xi,z}|(\mathbb{R}) d\mathcal{H}^{d-1}(z) < +\infty.$$

But, in fact, one has

$$\int_{\xi^\perp} |Du_{\xi,z}|(\mathbb{R}) d\mathcal{H}^{d-1}(z) = |\xi \cdot Du|$$

and it is enough to control this for a basis $\xi \in \{e_i : i = 1, \dots, d\}$ to obtain that $u \in BV$.

1D Slicing of BV or BD functions

In the same way:, $u \in BD$ if and only if for all $\xi \in \mathbb{S}^{d-1}$ and almost all $z \in \xi^\perp$,

$$u_{\xi,z} : s \mapsto \xi \cdot u(z + s\xi) \text{ is } BV, \quad \text{and}$$
$$|Eu| \approx \max_{\xi} \int_{\xi^\perp} |Du_{\xi,z}|(\mathbb{R}) d\mathcal{H}^{d-1}(z) < +\infty.$$

But, in fact, one has

$$\int_{\xi^\perp} |Du_{\xi,z}|(\mathbb{R}) d\mathcal{H}^{d-1}(z) = |\xi \cdot (Eu \cdot \xi)|$$

and it is enough to control this for a basis $\xi \in \{e_i : i = 1, \dots, d\}$ and the directions $\{e_i + e_j : 1 \leq i < j \leq d\}$ to obtain that $u \in BD$.

1D slicing of BV or BD functions

We recall indeed that [for u smooth]

$$\frac{d}{ds}(u(z+s\xi)) = \langle Du(z+s\xi), \xi\rangle$$

is controlled by $\int |\nabla u| dx$ (on average), but not (if u vectorial) by $\int |e(u)| dx$, while

$$\frac{d}{ds}(\xi \cdot u(z+s\xi)) = \langle {}^t\xi Du(z+s\xi), \xi\rangle = \langle {}^t\xi e(u)(z+s\xi), \xi\rangle$$

is a symmetric expression of Du and controlled (on average) by $\int |e(u)| dx$

Dal Maso's definition [JEMS, 2011]

Definition $u : \Omega \to \mathbb{R}^d$ (measurable) is in GBD if and only if for all (or a.e.) $\xi \in \mathbb{S}^{d-1}$ and a.e. $z \in \xi^\perp$,

- $u_{\xi,z} : s \mapsto \xi \cdot u(z + s\xi)$ is BV,
- $M_\xi := \int_{\xi^\perp} dz \left(\int_{\mathbb{R} \setminus J_{u_{\xi,z}}} |Du_{\xi,z}| + \sum_{s \in J_{u_{\xi,z}}} |u^+_{\xi,z}(s) - u^-_{\xi,z}(s)| \wedge 1 \right) < +\infty$,

and

$$\sup_{\xi \in \mathbb{S}^{d-1}} M_\xi < +\infty.$$

- DM shows that such u has a $(d-1)$-rectifiable jump set J_u and an approximate symmetrized gradient $e(u) \in L^1(\Omega)$ a.e.,
- Then, compactness and lower semicontinuity for minimizing sequences for $\mathcal{E}(u)$ (AC+Crismale);
- Then, weak minimizers are strong ($K = \overline{J_u}$, Conti-Focardi-Iurlano in 2D, AC-Conti-Iurlano in higher dimension, AC-Crismale with Dirichlet B.C.).

A simpler characterization

In DM's paper, one really needs that M_ξ is bounded for all ξ (or a dense set, but it is shown that $\xi \mapsto M_\xi$ is lsc).

Yet in a recent paper of Almi, Davoli, Kubin, Tasso (`arXiv:2410.23908`), a variant of *GBD* is introduced as the domain of the limit of non-local functionals (in the spirit of Bourgain-Brézis-Mironescu), where it is assumed that only

$$\int_{\mathbb{S}^{d-1}} M_\xi < +\infty,$$

and they raise the question: is this *GBD*?

A simpler characterization

A more natural question is: assume we know that for some basis $\{e_i : i = 1, \dots, d\}$,

$$\max_{1 \le i \le d} M_{e_i} \, , \max_{1 \le i < j \le d} M_{e_i + e_j} < +\infty$$

then is u in GBD?

The characterization of Almi *et al*. obviously implies that for a.e. orthogonal bases, this will hold. Hence if this characterizes GBD, then their variant is also GBD.

A simpler characterization

Theorem [C.-Crismale, 2025] Assume that there exists a basis $\{e_i : i = 1, \dots, d\}$ such that

$$\max_{1 \le i \le d} M_{e_i}, \max_{1 \le i < j \le d} M_{e_i + e_j} < +\infty$$

then $u \in GBD$.

(wlog, orthonormal basis)

Idea of proof

The proof is relatively simple: since the only tool we can rely on here is slicing, so nothing very fancy. We fix $\varepsilon > 0$ and show, for $V = \{e_i : i = 1, \dots, d\} \cup \{e_i + e_j : 1 \le i < j \le d\}$, and $Q = (0,1)^d$,

$$\varepsilon^{d-1} \int_Q \sum_{\xi \in V} \sum_{i \in \varepsilon \mathbb{Z}^d} \big(|\xi \cdot (u(\varepsilon y + i + \varepsilon \xi) - u(\varepsilon y + i))| \wedge 1\big) dy \le C \sum_{\xi \in V} M_\xi$$

Idea of proof

Indeed, given ξ,

$$\varepsilon^{d-1}\int_Q \sum_{i\in\varepsilon\mathbb{Z}^d} \big(|\xi\cdot(u(\underbrace{\varepsilon y+i}_{x}+\varepsilon\xi)-u(\varepsilon y+i))|\wedge 1\big)\,dy$$
$$=\varepsilon^{-1}\int_{\Omega\cap(\Omega-\varepsilon\xi)} \big(|\xi\cdot(u(x+\varepsilon\xi)-u(x))|\wedge 1\big)\,dx,$$
$$=\varepsilon^{-1}\int_{\xi^\perp}\int_{\Omega_{\xi,z}\cap(\Omega_{\xi,z}-\varepsilon)} \big(|u_{\xi,z}(s+\varepsilon)-u_{\xi,z}(s)|\wedge 1\big)|\xi|\,ds\,d\mathcal{H}^{d-1}(z).$$

If we let $\mu_{\xi,z}(I)=|Du_{\xi,z}|(I\setminus J_{u_{\xi,z}})+\sum_{s\in I\cap J_{u_{\xi,z}}}|u^+_{\xi,z}(s)-u^-_{\xi,z}(s)|\wedge 1$, we show that

$$|u_{\xi,z}(s+\varepsilon)-u_{\xi,z}(s)|\wedge 1\le \mu_{\xi,z}(s,s+\varepsilon)$$

for a.e. s.

Idea of proof

And then [using Fubini],

$$\int_{\Omega_{\xi,z}\cap(\Omega_{\xi,z}-\varepsilon)} |u_{\xi,z}(s+\varepsilon) - u_{\xi,z}(s)| \wedge 1 ds \le \int_{\Omega_{\xi,z}\cap(\Omega_{\xi,z}-\varepsilon)} \mu_{\xi,z}(s, s+\varepsilon) ds$$

$$\le \int_{\Omega_{\xi,z}\cap(\Omega_{\xi,z}-\varepsilon)} \chi_{\{s\le t\le s+\varepsilon\}} d\mu_{\xi,z}(t) ds$$

$$= \int_{\Omega_{\xi,z}\cap(\Omega_{\xi,z}-\varepsilon)} \chi_{\{t-\varepsilon\le s\le t\}} ds\, d\mu_{\xi,z}(t) \le \varepsilon \mu_{\xi,z}(\Omega_{\xi,z}).$$

Hence:

$$\varepsilon^{-1} \int_{\xi^\perp} \int_{\Omega_{\xi,z}\cap(\Omega_{\xi,z}-\varepsilon)} |u_{\xi,z}(s+\varepsilon) - u_{\xi,z}(s)| \wedge 1 ds \le \varepsilon^{-1} \int_{\xi^\perp} \varepsilon \mu_{\xi,z}(\Omega_{\xi,z}) \le M_\xi$$

Idea of proof

We have proved our claim:

$$\varepsilon^{d-1}\int_Q \sum_{\xi\in V}\sum_{i\in\varepsilon\mathbb{Z}^d}\big(|\xi\cdot(u(\varepsilon y+i+\varepsilon\xi)-u(\varepsilon y+i))|\wedge 1\big)dy \leq C\sum_{\xi\in V} M_\xi$$

so that for **many** $y\in Q$,

$$\varepsilon^{d-1}\sum_{\xi\in V}\sum_{i\in\varepsilon\mathbb{Z}^d}\big(|\xi\cdot(u(\varepsilon y+i+\varepsilon\xi)-u(\varepsilon y+i))|\wedge 1\big) \leq 2C\sum_{\xi\in V} M_\xi$$

and it turns out that one can select such y^ε such that for a.e. x,

$$\lim_{\varepsilon\to 0}\sum_{i\in\mathbb{Z}^d} u(\varepsilon y^\varepsilon+i)\prod_{j=1}^{d}\left(1-\frac{|x_j-i_j|}{\varepsilon}\right)^+ = u(x).$$

Idea of proof

We then define an approximating sequence u^ε as

- the multilinear interpolation of the values $u(\varepsilon y^\varepsilon + i)$ (defined in the previous slide) in the cubes such that

$$|\xi \cdot (u(\varepsilon y^\varepsilon + j + \varepsilon\xi) - u(\varepsilon y^\varepsilon + j))| \leq 1$$

  for all $\xi \in V$ and all pair of vertices $(j, j + \varepsilon\xi)$ of the cube;
- 0, else.

Then the construction guarantees that $u^\varepsilon \in SBD(\Omega)$ and $\mathcal{H}^{d-1}(J_{u^\varepsilon}) \leq C \sum_{\xi \in V} M_\xi$, in addition $u^\varepsilon \to u$ (in measure or almost everywhere). What about $\int_\Omega |e(u^\varepsilon)| dx$?

The control of the linear interpolates

Lemma Consider the unit cube $Q = [0,1]^d \subset \mathbb{R}^d$. Let $v \in (\mathbb{R}^d)^{\{0,1\}^d}$ be given at all vertices of Q such that $v_i(x+e_i) = v_i(x)$ for any $x \in \{0,1\}^d$ with $x_i = 0$ and $v_i(x+e_i+e_j) + v_j(x+e_i+e_j) = v_i(x) + v_j(x)$ for any $x \in \{0,1\}^d$ with $x_i = x_j = 0$.

For $x \in Q$, we also denote by $v(x)$ the multilinear interpolation of the values v at the vertices (affine on each $[x, x+e_i]$ for any $x \in Q$ with $x_i = 0$). Then $e(v) = 0$ in Q (so that, in fact, v is affine with skew-symmetric gradient).

Corollary: there exists $C > 0$ such that

$$\int_Q |e(v)|dx \le C \Bigg(\sum_{i=1}^{d} \sum_{\substack{x\in\{0,1\}^d \\ x_i=0}} |v_i(x+e_i) - v_i(x)| \\ + \sum_{i,j=1}^{d} \sum_{\substack{x\in\{0,1\}^d \\ x_i=x_j=0}} |v_i(x+e_i+e_j) + v_j(x+e_i+e_j) - v_i(x) - v_j(x)| \Bigg)$$

Conclusion

We end up with $u^\varepsilon \to u$ such that $u^\varepsilon \in SBV$ and

$$\int_\Omega |e(u^\varepsilon)| + \mathcal{H}^{d-1}(J_{u^\varepsilon}) \leq C \sum_{\xi \in V} M_\xi < +\infty$$

$\to$ (compactness and) lower-semicontinuity ensures the limit is GBD. (AC+Crismale, 2023/25.) $\square$