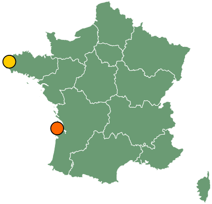

A piston to counteract diffusion: The influence of an inward-shifting boundary on the heat equation in half-space

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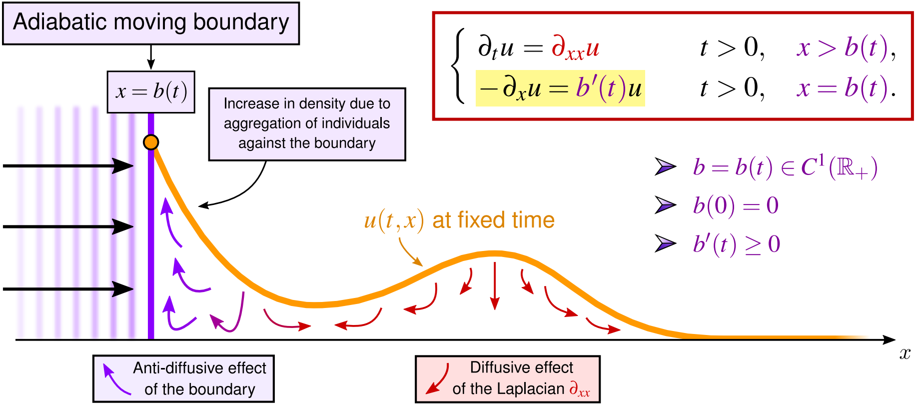
12ième Biennale Française des Mathématiques Appliquées et Industrielles

June 2–6, 2025



$$\begin{cases} \partial_t u = \partial_{xx} u & t > 0, \quad x > b(t), \\ -\partial_x u = b'(t) u & t > 0, \quad x = b(t). \end{cases}$$

- $b = b(t) \in C^1(\mathbb{R}_+)$
- $b(0) = 0$
- $b'(t) \geq 0$

(means no individual leakage)

- Mass conservation, that is: $\quad M(t) := \int_{x=b(t)}^{\infty} u(t,x)dx \equiv M|_{t=0}$,

  requires non-autonomous Robin-type boundary condition at $x = b(t)$:

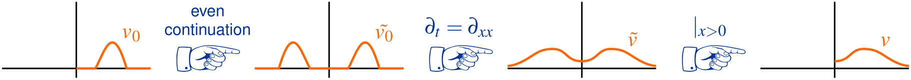
$$0 = M'(t) = -b'(t)u(t,b(t)) + \int_{x=b(t)}^{\infty} \partial_{xx} u(t,x)dx = -b'(t)u(t,b(t)) - \partial_x u(t,b(t))$$

(consistent with homogeneous Neumann if $b \equiv 0$...)

$$\begin{cases} \partial_t u = \partial_{xx} u & t > 0, \quad x > b(t), \\ -\partial_x u = b'(t) u & t > 0, \quad x = b(t). \end{cases} \xrightarrow{v(t,x) := u(t, x + b(t))} \begin{cases} \partial_t v = \partial_{xx} v + b'(t)\partial_x v & t > 0, \quad x > 0, \\ -\partial_x v = b'(t) v & t > 0, \quad x = 0. \end{cases}$$

- Simplest case: $b \equiv 0$ $\begin{cases} \partial_t v = \partial_{xx} v & t > 0, \quad x > 0, \\ -\partial_x v = 0 & t > 0, \quad x = 0. \end{cases}$ Heat equation in $\{x > 0\}$ with homogeneous Neumann BC

- Solve by continuation argument:



$$v(t,x) = \int_{z=0}^{\infty} \frac{1}{\sqrt{4\pi t}} \left(e^{-\frac{(x-z)^2}{4t}} + e^{-\frac{(x+z)^2}{4t}} \right) v_0(z)\, dz \quad \Rightarrow \quad v(t,0) \approx \frac{c}{2\sqrt{t}}$$

- Magnitude of the mean position of a reflected Brownian particle: $\mathbb{E}(X_t) = \int_{z=0}^{\infty} z v(t,z) dz$

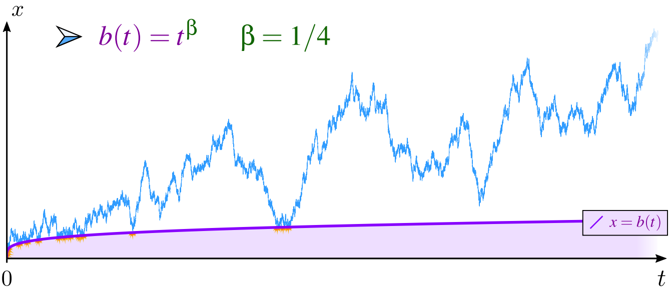
$$\frac{d}{dt}\mathbb{E}(X_t) = \int_{z=0}^{\infty} z v_{zz} = [z v_z]_{z=0}^{\infty} - \int_{z=0}^{\infty} v_z = v(t,0) \quad \Rightarrow \quad \boxed{\mathbb{E}(X_t) \approx c\sqrt{t}}$$

- Suggests that the algebraic forms $b(t) = t^{\beta}$ with $\beta > 0$ are good candidates for the boundary motion... } in particular, near $\beta = 1/2$

Brownian motion reflected at x=b(t)

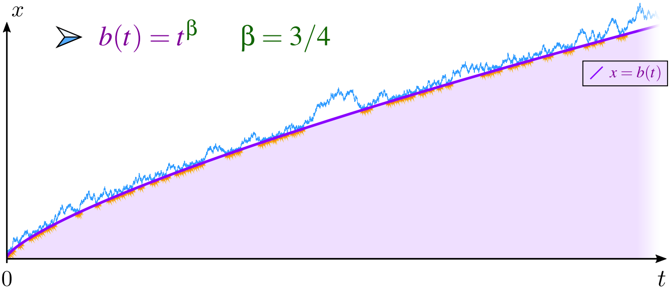
$$b(t) \ll \sqrt{t}$$

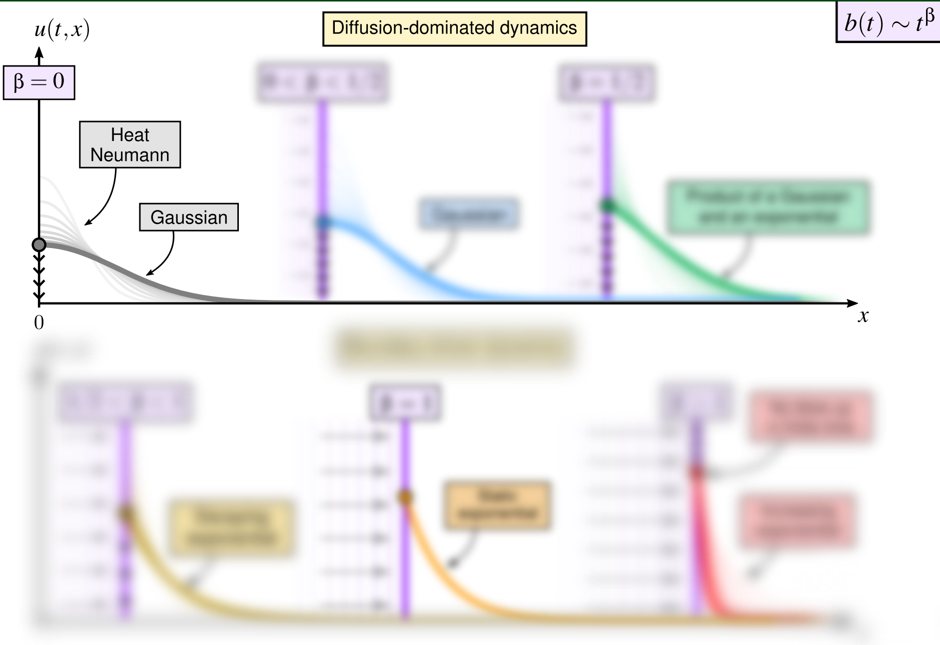
Diffusion-dominated dynamics



$$b(t) \gg \sqrt{t}$$

Boundary-driven dynamics




Diffusion-dominated dynamics
$b(t) \sim t^{\beta}$
$u(t,x)$
$\beta = 0$
Heat
Neumann
Gaussian
0
x

- Starting easy: $b(t) = t$

$$\begin{cases} \partial_t v = \partial_{xx} v + \partial_x v & t > 0, \quad x > 0, \\ -\partial_x v = v & t > 0, \quad x = 0. \end{cases}$$

Autonomous advective heat equation in $\{x > 0\}$ with Robin BC

- Continuation arguments similar to the Neumann case also apply here.

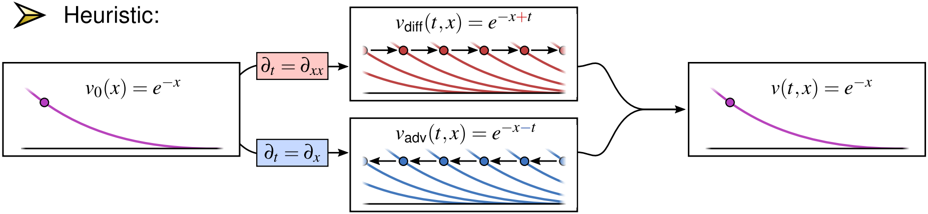
Theorem (S.T., M.Zhang – 2025)

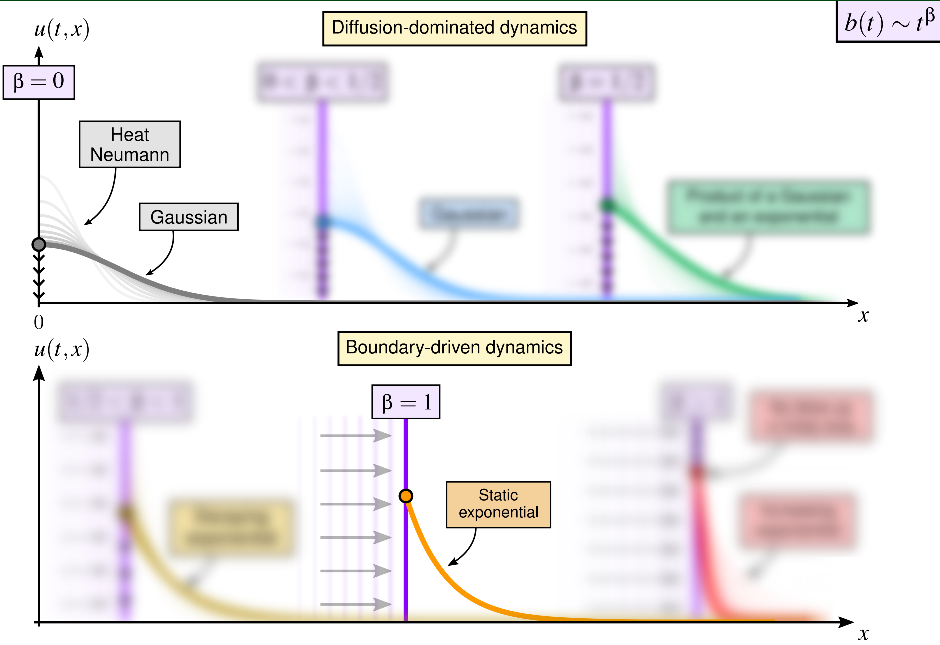
- **Explicit** fundamental solution: $v(t,x) = \int_{z=0}^{\infty} H(t,x,z) v_0(z) dz.$
- Convergence toward stationary exponential profile at exponential rate:

$$\|v(t,x) - (\textstyle\int v_0) e^{-x}\|_{L^p_x(\mathbb{R}_+)} \leq \ell e^{-kt}, \qquad \text{for any } 1 \leq p \leq \infty.$$

The moving boundary perfectly compensates the diffusion!

- Heuristic:

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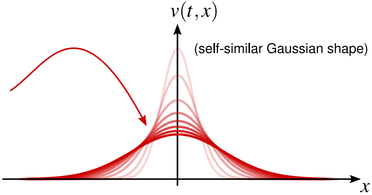
$u(t,x)$
$\beta = 1$
Static
exponential
x

- Heat equation: $\partial_t v = \partial_{xx} v \quad \text{in } \mathbb{R}$

  - Fundamental solution: $v(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$



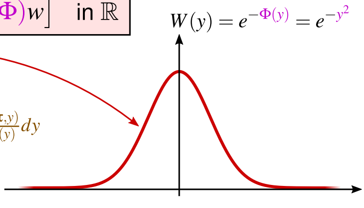
Want to capture the **shape of the solution**

(to be "mass consistent")

Self-similar rescaling*: $v(t,x) = f(t)\,w(g(t), f(t)x)$ $\quad \tau := g(t) \quad y := f(t)x$

- Fokker-Planck equation: $\partial_\tau w = \partial_y \left[\partial_y w + (\partial_y \Phi) w\right] \quad \text{in } \mathbb{R}$

  - Steady state: $W(y) = e^{-\Phi(y)}$



- Proof of convergence *via* entropy decay: $H(\tau) := \int_{\mathbb{R}} w(\tau,y) \log \frac{w(\tau,y)}{W(y)} dy$

$$\frac{d}{d\tau} H(\tau) \leq -2H(\tau) \Rightarrow H(\tau) \lesssim e^{-2\tau}$$

$$\Rightarrow \|w(\tau,y) - W(y)\|_{L^1_y(\mathbb{R})} \lesssim e^{-\tau} = \frac{1}{\sqrt{t}}.$$

(Grönwall lemma)

(log-Sobolev inequality) (Csiszár–Kullback inequality)

* In this case, $f(t) = \frac{\alpha}{\sqrt{t}}$, $g(t) = \gamma \log(t)$, and $\partial_y \Phi(y) = 2y$.

- Piston diffusion problem:

$$\begin{cases} \partial_t v = \partial_{xx} v + b'(t)\partial_x v & t>0, \quad x>0, \\ -\partial_x v = b'(t) v & t>0, \quad x=0. \end{cases}$$

$$b(t) = (1+t)^\beta \qquad 0 \le \beta \le 1/2$$

Self-similar rescaling: $v(t,x) = \frac{1}{\sqrt{1+t}} w(\tau, y) \qquad \tau := \frac{1}{2}\log(1+t) \qquad y := \frac{x}{\sqrt{1+t}}$

- Fokker-Planck equation:

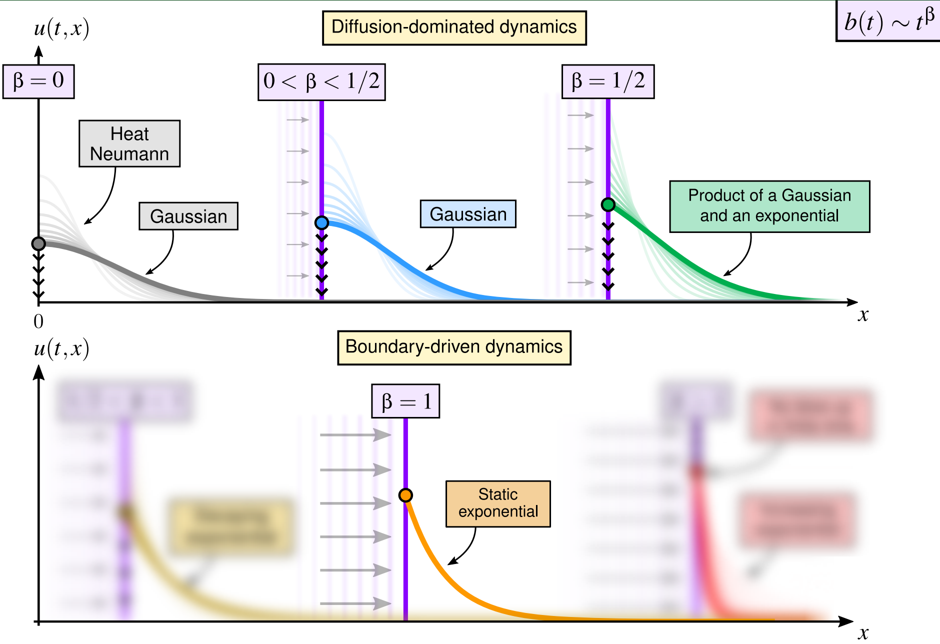
$$\begin{cases} \partial_\tau w = \partial_y \left[2\partial_y w + (y + \psi(\tau)) w \right] & \tau>0, \quad y>0, \\ -\partial_y w = \psi(\tau) w & \tau>0, \quad y=0. \end{cases}$$

- Where $\psi(\tau) \begin{cases} \xrightarrow{\tau\to\infty} 0 & \text{if } 0 \le \beta < 1/2, \\ \equiv 1 & \text{if } \beta = 1/2. \end{cases}$
- Proof of convergence *via* entropy decay
- Steady state: $W(y) = \begin{cases} e^{-\frac{y^2}{4}} & \text{if } 0 \le \beta < 1/2, \\ e^{-\frac{y^2}{4} - \frac{y}{2}} & \text{if } \beta = 1/2. \end{cases}$

Theorem (S.T., M.Zhang – 2025)

Assume $0 \le \beta \le 1/2$, $b(t) = (1+t)^\beta$, and v_0 nonnegative, bounded and compactly supported, then for any $t>0$,

$$\left\| v(t,x) - \frac{1}{\sqrt{1+t}} W\left(\frac{x}{\sqrt{1+t}}\right) \right\|_{L^1_x(\mathbb{R}_+)} \lesssim \begin{cases} \frac{1}{(1+t)^{\frac{1}{4}-\frac{\beta}{2}}} & \text{if } 0 \le \beta < 1/2, \\ \frac{1}{\sqrt{1+t}} & \text{if } \beta = 1/2. \end{cases}$$


$b(t) \sim t^{\beta}$
Diffusion-dominated dynamics
$u(t,x)$
$\beta = 0$
Heat
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$0 < \beta < 1/2$
Gaussian
$\beta = 1/2$
Product of a Gaussian
and an exponential
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Boundary-driven dynamics
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Static
exponential
x

$$b(t) = (1+t)^{\beta} \qquad \beta > 1/2$$

- Piston diffusion problem:

$$\begin{cases} \partial_t v = \partial_{xx} v + b'(t)\partial_x v & t>0, \quad x>0, \\ -\partial_x v = b'(t) v & t>0, \quad x=0. \end{cases}$$

Self-similar rescaling: $v(t,x) = \frac{b'(t)}{\beta} w(\tau, y) \qquad \tau := \int_{s=0}^{t} \left[\frac{b'(s)}{\beta}\right]^2 ds \qquad y = \frac{b'(t)}{\beta} x$

- Fokker-Planck equation:

$$\begin{cases} \partial_\tau w = \partial_y \left[\partial_y w + (\beta + \eta(\tau) y) w\right] & \tau>0, \quad y>0, \\ -\partial_y w = \beta w & \tau>0, \quad y=0. \end{cases}$$

- Where $\eta(\tau) \xrightarrow{\tau\to\infty} 0$
- Steady state: $W(y) = e^{-\beta y}$

- Proof of convergence *via* semigroup approach:
  we see the term $\partial_y(\eta(\tau) y\, w)$ as a perturbative source term to apply Duhamel's principle:

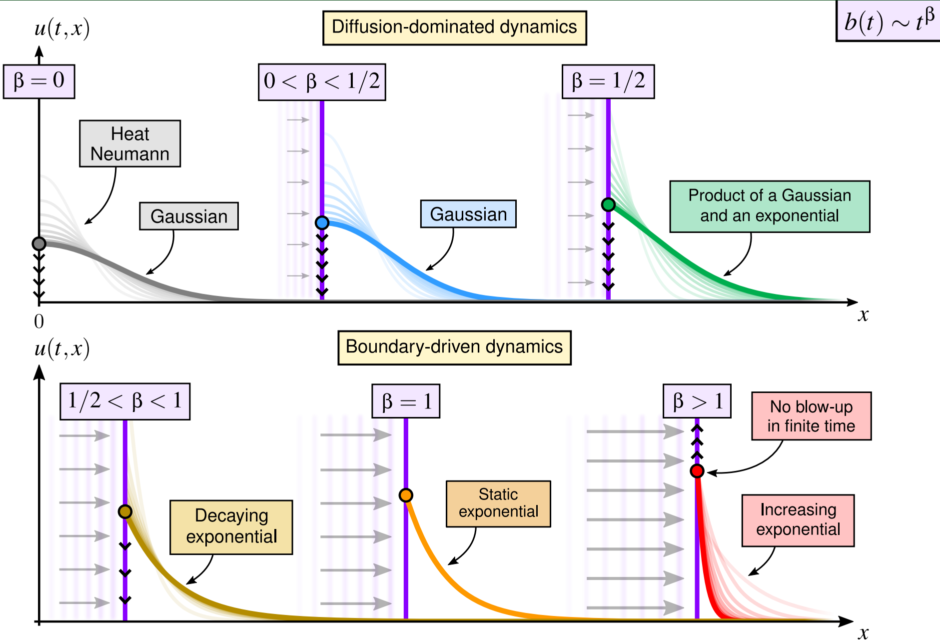
$$w(\tau, y) = \mathcal{S}_\tau w_0(y) + \int_{s=0}^{\tau} \mathcal{S}_{\tau-s}\left[\partial_y(\eta(s) y\, w)\right] ds$$

As $\mathcal{S}_\tau w_0 \to W$ exponentially fast, we show that the convolution term vanishes as $\tau \to \infty$.
(finiteness of the first moment of w is crucial, and becomes an obstruction when $\beta > 1$)

Theorem (S.T., M.Zhang – 2025)

Assume $1/2 < \beta \leq 1$, $b(t) = (1+t)^{\beta}$, and v_0 nonnegative, bounded and compactly supported, then for any $t > 1$,

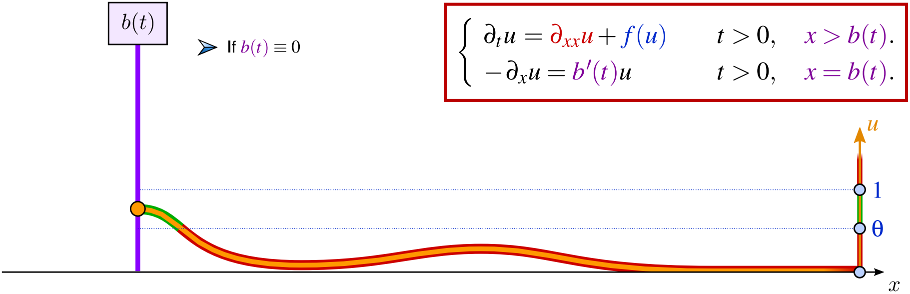
$$\left\| v(t,x) - \frac{b'(t)}{\beta} W\left(\frac{b'(t)}{\beta} x\right) \right\|_{L^1_x(\mathbb{R}_+)} \lesssim \frac{\log(1+t)}{(1+t)^{\beta - \frac{1}{2}}}.$$



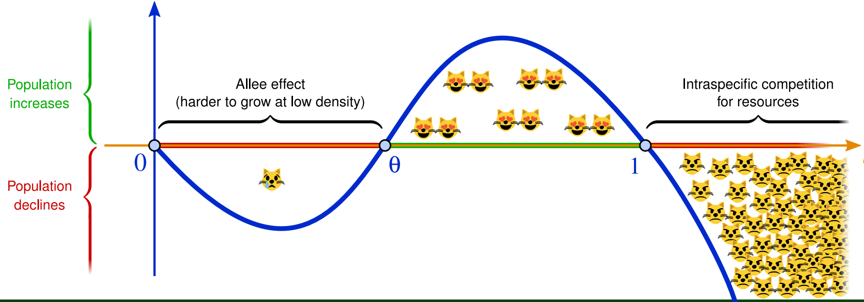
$b(t) \sim t^{\beta}$
Diffusion-dominated dynamics
$u(t,x)$
$\beta = 0$
Heat Neumann
Gaussian
$0 < \beta < 1/2$
Gaussian
$\beta = 1/2$
Product of a Gaussian and an exponential
0
x
Boundary-driven dynamics
$u(t,x)$
$1/2 < \beta < 1$
Decaying exponential
$\beta = 1$
Static exponential
$\beta > 1$
No blow-up in finite time
Increasing exponential
x

Perspectives: interplay with an Allee effect

$$\begin{cases} \partial_t u = \partial_{xx} u + f(u) & t > 0, \quad x > b(t). \\ -\partial_x u = b'(t) u & t > 0, \quad x = b(t). \end{cases}$$

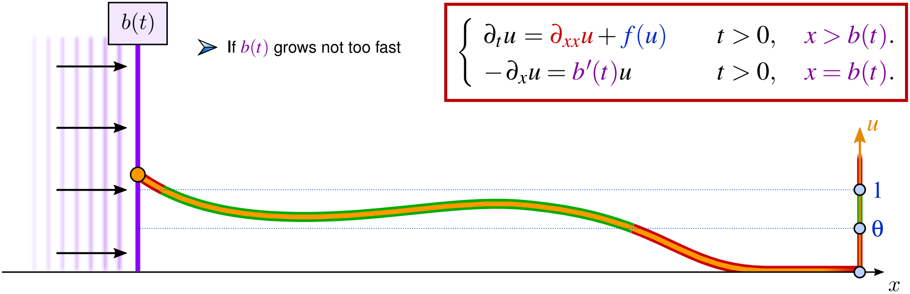


Bistable : $f(u) = u(1-u)(u-\theta)$

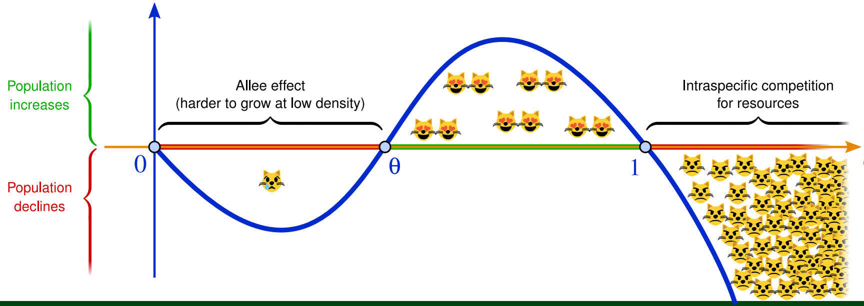


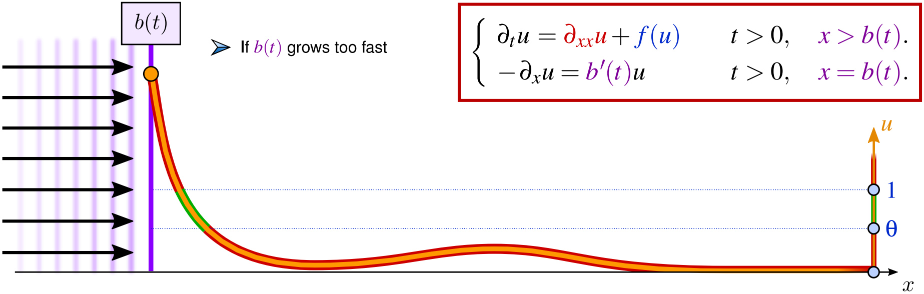
➢ If $b(t)$ grows not too fast

$$\begin{cases} \partial_t u = \partial_{xx} u + f(u) & t > 0, \quad x > b(t). \\ -\partial_x u = b'(t) u & t > 0, \quad x = b(t). \end{cases}$$



Bistable : $f(u) = u(1-u)(u-\theta)$





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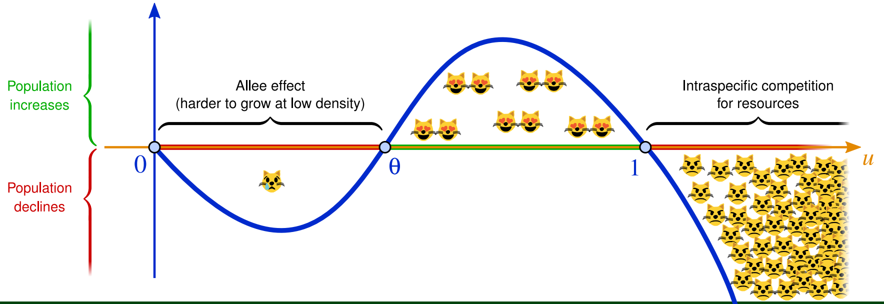

Thanks for your attention!

Tréton and Zhang
A piston to counteract diffusion: the influence of an inward-shifting boundary on the heat equation in half-space
arXiv:2505.03304 (2025)

Vázquez
Asymptotic behaviour methods for the Heat Equation. Convergence to the Gaussian
arXiv.1706.10034

Allwright
Exact solutions and critical behaviour for a linear growth-diffusion equation on a time-dependent domain
Proceedings of the Edinburgh Mathematical Society (2022)

Allwright
Reaction–diffusion on a time-dependent interval: Refining the notion of 'critical length'
Communications in Contemporary Mathematics (2023)

Lepoutre, Meunier and Muller
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