Une équation de Fisher-KPP pour une population structurée en espace et en phénotype

travail avec Luca Rossi

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SMAI

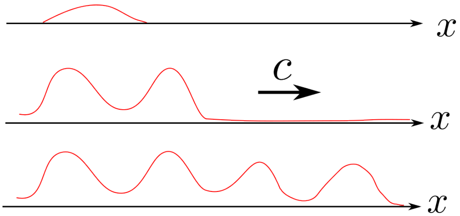
Fisher-KPP equation

$u(t,x)$: density of individuals $\qquad x \in \mathbb{R}$

$$\begin{array}{cccccc} & & \text{movements} & + & \text{demography}-\text{competition} \\ \partial_t u(t,x) & = & \Delta u & + & (r(x)-u)u \end{array}$$

Heterogeneous **periodic** environment:

$u(t,x) \rightarrow$



Persistence and principal eigenvalue

Close to extinction ($\sup u(t,\cdot)$ small):

$$\partial_t u(t,x,\theta) \simeq (\Delta + r)u.$$

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If $u(0,x) = \varepsilon\varphi(x)$:

$$\partial_t u \simeq \lambda u \quad \rightarrow \quad u(t,x) \simeq e^{\lambda t} u(0,x).$$

$$\lambda \leq 0 : \text{extinction}$$

$$\lambda > 0 : \text{persistence}$$

Cantrell, Cosner 1989; Berestycki, Hamel, Roques 2005

Add a phenotype variable

$u(t,x) \to u(t,x,\theta), \qquad x \in \mathbb{R}^N,\ \theta \in \mathbb{R}^P$

$$\begin{array}{ccccccc} & & \text{movements} & + & \text{mutations} & + & \text{demography}-\text{compet.} \\ \partial_t u(t,x,\theta) & = & \Delta_x u & + & \Delta_\theta u & + & (r(x,\theta)-\rho)u \end{array}$$

$$\begin{aligned} \rho(t,x) &= \int_{\mathbb{R}^P} u(t,x,\theta)\,\mathrm{d}\theta \\ &= \text{total population at time } t \text{ and position } x. \end{aligned}$$

$$r(x,\theta) = \text{fitness of phenotype } \theta \text{ at position } x.$$

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Prévost 2004; Champagnat, Méléard 2007

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Goal: Persistence or extinction? $\rightarrow$ principal eigenvalue

$$\liminf_{t\to+\infty} \sup_{x\in\mathbb{R}^N} \rho(t,x) > 0 \quad \textit{vs} \quad \limsup_{t\to+\infty} \sup_{x\in\mathbb{R}^N} \rho(t,x) = 0.$$

Persistence and principal eigenvalue

B., Rossi, *Reaction–diffusion model for a population structured in phenotype and space: I. Criterion for persistence*, Nonlinearity, 2025.

Generalised principal eigenvalue

$r \in L^\infty_{loc}(\mathbb{R}^N \times \mathbb{R}^P)$ x-periodic, globally bounded above.

Don't require $r(x,\theta) \to -\infty$ as $\|\theta\| \to +\infty$!

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Generalised principal eigenvalue [Berestycki, Rossi 2009, 2015]

Let

$$\lambda := \inf \{\lambda' \in \mathbb{R} \ / \ \exists \phi > 0, (\Delta_{x,\theta} + r(x,\theta))\phi \leq \lambda'\phi\}.$$

Then $\lambda > -\infty$ and there exists $\varphi(x,\theta) > 0$ s.t. $(\Delta_{x,\theta} + r(x,\theta))\varphi = \lambda\varphi$.

$\lambda < 0$: extinction

$\lambda > 0$: persistence

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Then $\lambda > -\infty$ and there exists $\varphi(x,\theta) > 0$ s.t. $(\Delta_{x,\theta} + r(x,\theta))\varphi = \lambda\varphi$.

$$\lambda < 0 : \text{extinction}$$

$$\lambda > 0 : \text{persistence}$$

But:

$\to$ for all $\lambda' \geq \lambda$, there is $\varphi_{\lambda'} > 0$ such that $(\Delta_{x,\theta} + r(x,\theta))\varphi_{\lambda'} = \lambda'\varphi_{\lambda'}$

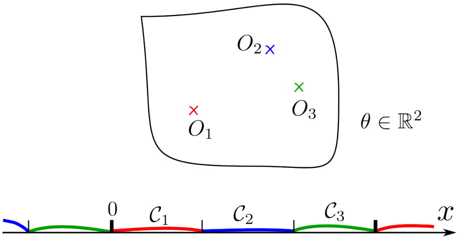
$\to$ φ **might go to** $+\infty$ **as** $\|\theta\| \to +\infty$

Optimisation of the ability of persistence

B., Rossi, *Reaction–diffusion model for a population structured in phenotype and space. II. Optimisation of the ability of persistence*, ongoing.

A patchwork landscape

u = population of pathogens;
living on *fields* $\mathcal{C}_i$ (periodic); O_i = optimal phenotype on field i



Fisher Geometric model:

$$r(x,\theta) := \begin{cases} \chi(\|\theta - O_1\|) & \text{if } x \in \mathcal{C}_1, \\ \vdots & \\ \chi(\|\theta - O_K\|) & \text{if } x \in \mathcal{C}_K, \end{cases}$$

with $\chi \in \mathcal{C}^2(\mathbb{R}_+)$ decreasing and $\inf \chi \leq 0$.

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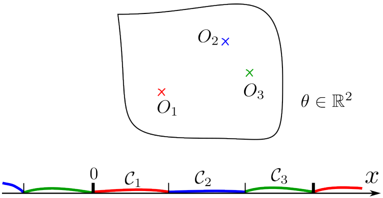
Idea: the closer the optima O_i, the more favourable the environment



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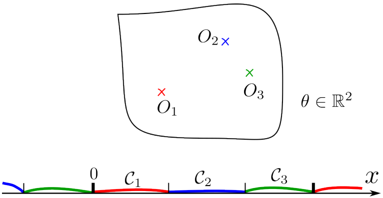

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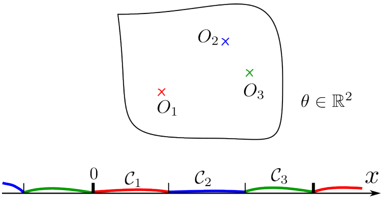

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→ Given two configurations $O_1, \ldots, O_K$ and $\hat{O}_1, \ldots, \hat{O}_K$ satisfying

$$\|\hat{O}_i - \hat{O}_j\| \leq \|O_i - O_j\| \quad \text{for all } i,j = 2,\ldots,K,$$

is it true that $\lambda[\hat{O}_i] \geq \lambda[O_i]$?

Extra assumptions on χ

There exists $\varphi(x,\theta) > 0$ such that $(\Delta_{x,\theta} + r(x,\theta))\varphi = \lambda\varphi$.

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Proposition [B., Rossi 25+] If either of the following holds:

- $\to$ $\inf r = -\infty$,
- $\to$ $\theta \in \mathbb{R}^P$ with $P = 1$ or $P = 2$,
- $\to$ $\chi(R) - \inf \chi > A - R^2$
  where $A > P$ depends on the dimension P and the fields $\mathcal{C}_i$;

THEN $\lambda > \inf r$, which implies:

φ decays exponentially as $\|\theta\| \to +\infty$

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Theorem [B., Rossi, 25+] Let $\gamma : [0,1] \to \mathbb{R}^P$ be a $\mathcal{C}^1$ curve, with $\gamma' \neq 0$ on $[0,1]$, such that $s \mapsto \|\gamma(s) - O_i\|$ is decreasing for $i = 2, \dots, K$.

If $O_1 = \gamma(0)$ and $\hat{O}_1 = \gamma(1)$ then

$$\lambda[\hat{O}_1] > \lambda[O_1].$$

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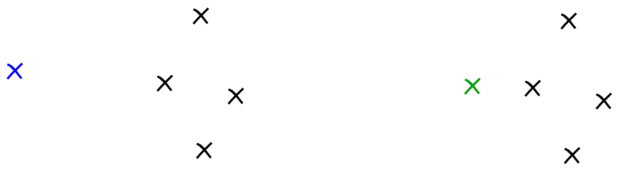
Two configurations:

cross $= O_i \in \mathbb{R}^2$



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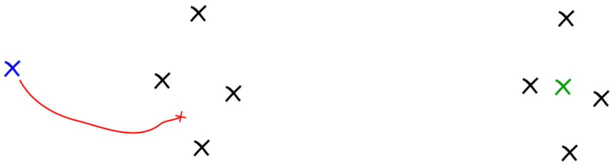

Dilatation

Theorem (B., Rossi 25+)

Let $O_1, \ldots, O_K \in \mathbb{R}^P$ be distinct, satisfying a convexity condition.

$\rightarrow$ $a \mapsto \lambda[r[aO_1, \ldots, aO_K]]$ $(a \geq 0)$ is strictly decreasing.

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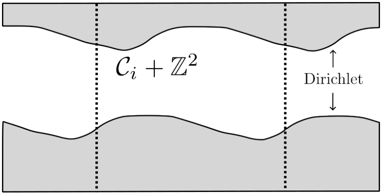
- $\rightarrow$ $a \mapsto \lambda[r[aO_1, \ldots, aO_K]]$ $(a \geq 0)$ is strictly decreasing.
- $\rightarrow$ If $\inf r = -\infty$ and for all i, $\mathcal{C}_i + \mathbb{Z}^N$ is $C^{1,1}$,

$$\lim_{a \to +\infty} \lambda[r[aO_1, \ldots, aO_K]] = \max_{i=1,\ldots,K} \lambda[i],$$

where $\lambda[i]$ is defined as $\lambda[r]$ but with Dirichlet on $\partial(\mathcal{C}_i + \mathbb{Z}^N)$



- $\rightarrow$ Analogous result if $\inf r > -\infty$, without regularity assumption, with another definition of $\lambda[i]$.

Open problems

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Spreading properties

- $\rightarrow$ On persistence, is there a spreading speed?

  Yes if bounded phenotype space
- $\rightarrow$ (non-)Existence/uniqueness of stationary states and pulsating traveling waves?
- $\rightarrow$ On persistence, does the solution converge to a pulsating traveling wave?

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→ On persistence, does the solution converge to a pulsating traveling wave?

About the model

→ Given two configurations $O_1, \dots, O_K$ and $\hat{O}_1, \dots, \hat{O}_K$ satisfying $\|\hat{O}_i - \hat{O}_j\| \leq \|O_i - O_j\|$ for all $i, j = 2, \dots, K$, is it true that $\lambda[\hat{O}_i] \geq \lambda[O_i]$?

→ What is the effect of adding a new field?

→ What is the effect of the shapes of the fields $\mathcal{C}_i$?

Merci !

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