An alternative Epicardial Model for the inverse problem of cardiac electrophysiology

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Carcans

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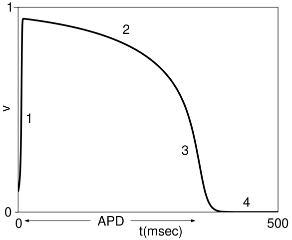
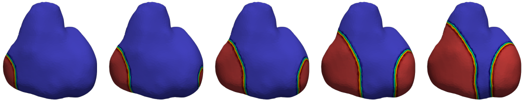
Inria

- **Context**
- A 3D heuristic model
- Comparing three inverse problems

Heart electrophysiology

- Cardiac rhythm disorders: pathologies of heart's electrical activity
- Electrical activation: gives the signal that precedes the contraction
- Action potential
- Activation maps
- Goal: detect pathologies from torso potentials, ECG imaging. Very ill posed inverse problem

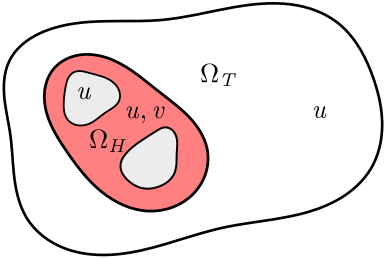




The reference bidomain model

Potential $u \in H^1(\Omega_T \cup \Omega_H)$, transmembrane voltage $v \in H^1(\Omega_H)$.

$$
\begin{aligned}
&\mathrm{div}(\sigma_i \nabla (u+v)) = \partial_t v + I_{ion}(v,t,h) && \Omega_H \times [0,T] \\
&\mathrm{div}((\sigma_e + \sigma_i)\nabla u) = -\mathrm{div}(\sigma_i \nabla v) && \Omega_H \times [0,T] \\
&\mathrm{div}(\sigma_T \nabla u) = 0 && \Omega_T \times [0,T] \\
&\sigma_i \nabla (u+v) \cdot n = 0 && \partial\Omega_H \times [0,T] \\
&\sigma_T \nabla u \cdot n = 0 && \partial\Omega_T \times [0,T]
\end{aligned}
$$

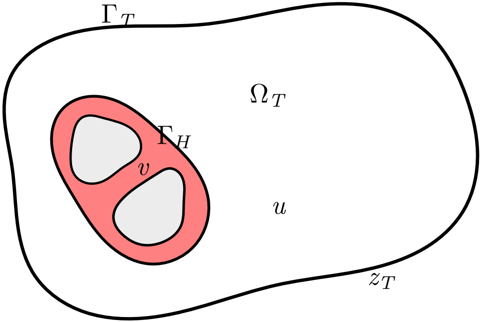


The inverse problem of cardiac electrophysiology

Find u and/or v that satisfy (part of) the bidomain equations, matching torso data z_T.
Find the activation map.

$$
\begin{aligned}
&\mathrm{div}((\sigma_e+\sigma_i)\nabla u) = -\mathrm{div}(\sigma_i\nabla v) && \Omega_H \\
&\mathrm{div}(\sigma_T\nabla u) = 0 && \Omega_T \\
&\sigma_i\nabla(u+v)\cdot n = 0 && \partial\Omega_H \\
&\sigma_T\nabla u\cdot n = 0 && \partial\Omega_T
\end{aligned}
$$

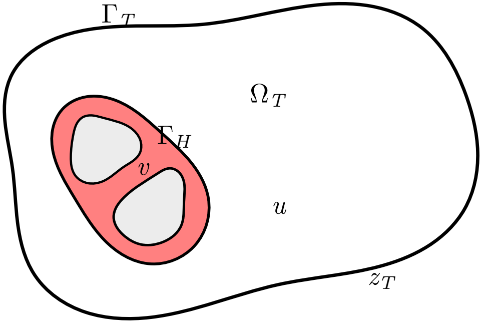
- In the heart surface: u: uniqueness (up to a constant) but non existence, v can be found up to a constant
- In the heart volume: infinite-dimensional kernel for v



The usual (static) Cauchy problem

Find u from data z_T

$-\Delta u = 0$	Ω_T
$u = z_T$	Γ_T
$\partial_n u = 0$	Γ_T



2D Depth-averaged model

$$\sigma_\ell h \frac{\partial^2}{\partial s^2}\bar{u} + \sigma_T \frac{\partial u_T}{\partial n}|_{\Gamma_H} + \sigma_B \frac{\partial u_B}{\partial n}|_{\Gamma_H} + \sigma_{i\ell} h \frac{\partial^2}{\partial s^2}\bar{v} = 0 \quad \Gamma_H$$

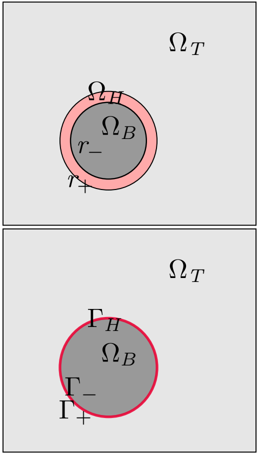
$$\mathrm{div}(\sigma_T \nabla u_T) = 0 \quad \Omega_T$$

$$\mathrm{div}(\sigma_B \nabla u_B) = 0 \quad \Omega_B$$

$$\sigma_T \partial_n u = 0 \quad \Gamma_T$$

$$\sigma_{ep} \frac{u_T - \bar{u}}{h} = (1-\alpha)\sigma_T \partial_n u_T \quad \Gamma_H$$

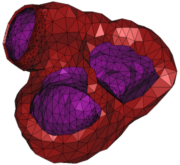
$$\sigma_{ep} \frac{u_B - \bar{u}}{h} = \alpha \sigma_B \partial_n u_B \quad \Gamma_H$$

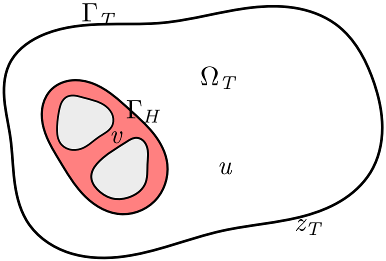


Emma Lagracie, Yves Bourgault, Yves Coudière and Lisl Weynans, *A depth-averaged heart model for the inverse problem of cardiac electrophysiology*, 2025, Inverse Problems

Observations

- Endocardial signals are not observable
- Complex 3D shape of the heart

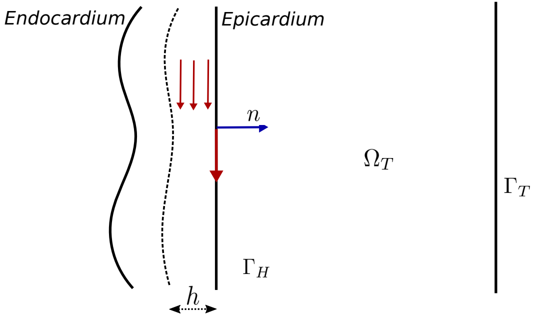




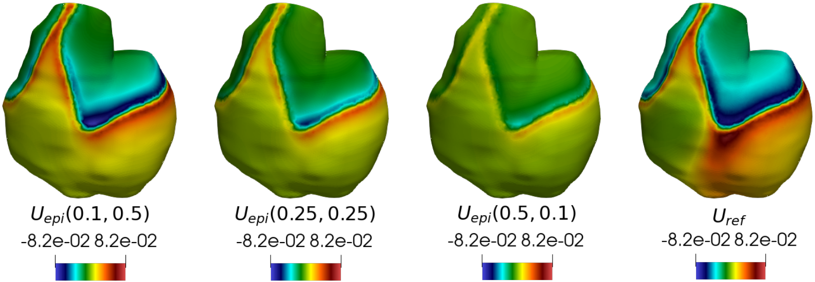
Heuristic model: the Epicardial model

$$\begin{aligned}
&\mathrm{div}_S(h(x)(\sigma_i^S + \sigma_e^S)\nabla_S \bar{u})) + \sigma_T \nabla u \cdot n \\
&= -\mathrm{div}_S(h(x)(\sigma_i^S \nabla_S v)) && \Gamma_H \\
&\mathrm{div}(\sigma_T \nabla u) = 0 && \Omega_T
\end{aligned}$$

$$\begin{aligned}
&\sigma_T \nabla u \cdot n = 0 && \Gamma_T \\
&\sigma^n \frac{u - \bar{u}}{\alpha h} = \sigma_T \nabla u \cdot n && \Gamma_H
\end{aligned}$$



Numerical validation and calibration



Three minimization problems

$$\min J(u,v) = \frac{1}{2}\int_{\Gamma_T} |u - z_T|^2 + \frac{\varepsilon}{2}\int_{\Gamma_H} |\nabla v|^2 + \frac{\varepsilon_{\text{inv}}}{2}\int_{\Gamma_H} |v|^2$$

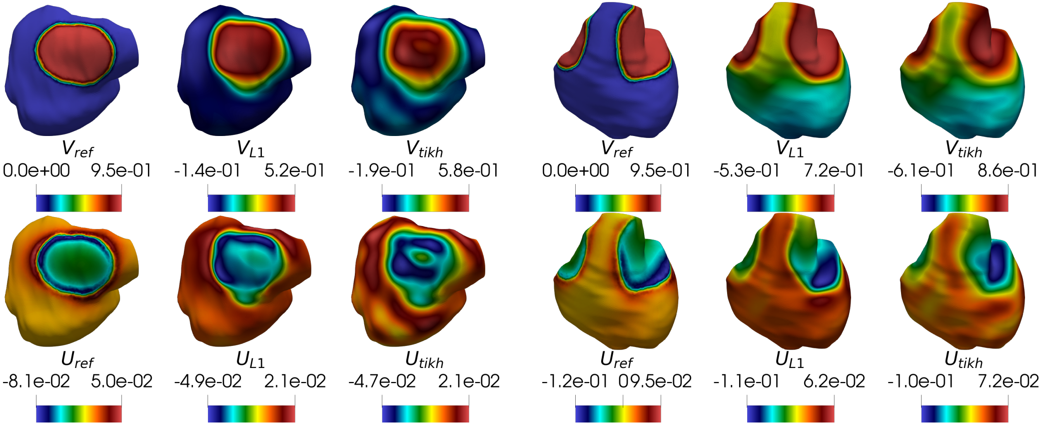
$$\min J_{pp}(u) = \frac{1}{2}\int_{\Gamma_T} |u - z_T|^2 + \frac{\varepsilon}{2}\|\nabla u\|^2_{L^2(\Gamma_H)}$$

$$\begin{array}{ll} -\Delta u = 0 & \Omega_T \\ u = u_H & \Gamma_H \\ \partial_n u = 0 & \Gamma_T \end{array}$$

$$\min J_{TV}(u,v) = \int_{\Gamma_T} |u - z_T| + \varepsilon\int_{\Gamma_H} |\nabla v| + \frac{\varepsilon_{\text{inv}}}{2}\int_{\Gamma_H} |v|^2$$

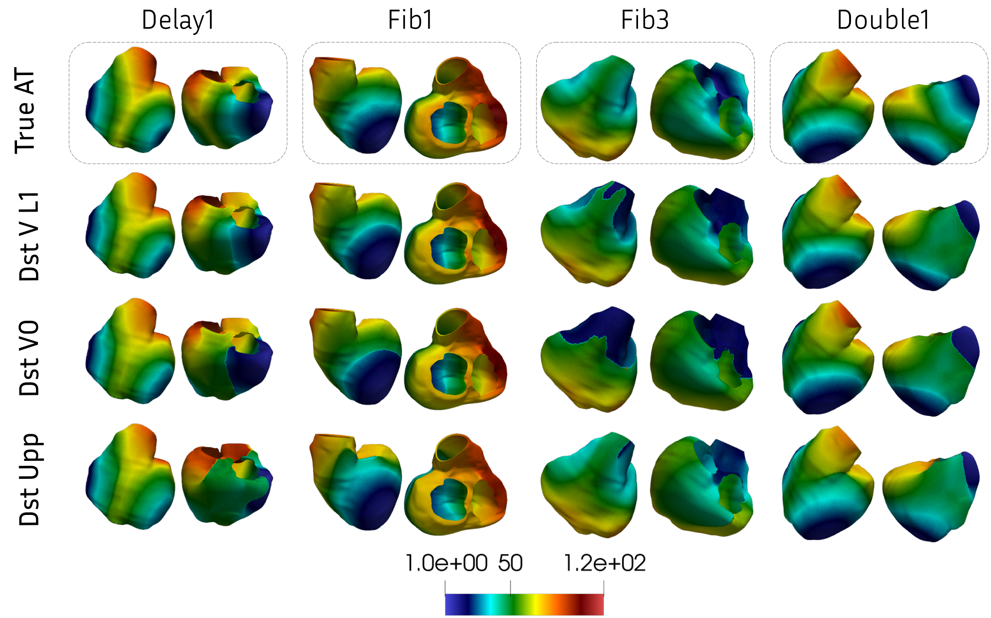
$$\begin{array}{ll} \text{div}_S(h(x)(\sigma_i^S + \sigma_e^S)\nabla_S \bar{u})) + \sigma_T \nabla u \cdot n & \\ = -\text{div}_S(h(x)(\sigma_i^S \nabla_S v)) & \Gamma_H \\ \text{div}(\sigma_T \nabla u) = 0 & \Omega_T \end{array}$$

Fixed time reconstructions



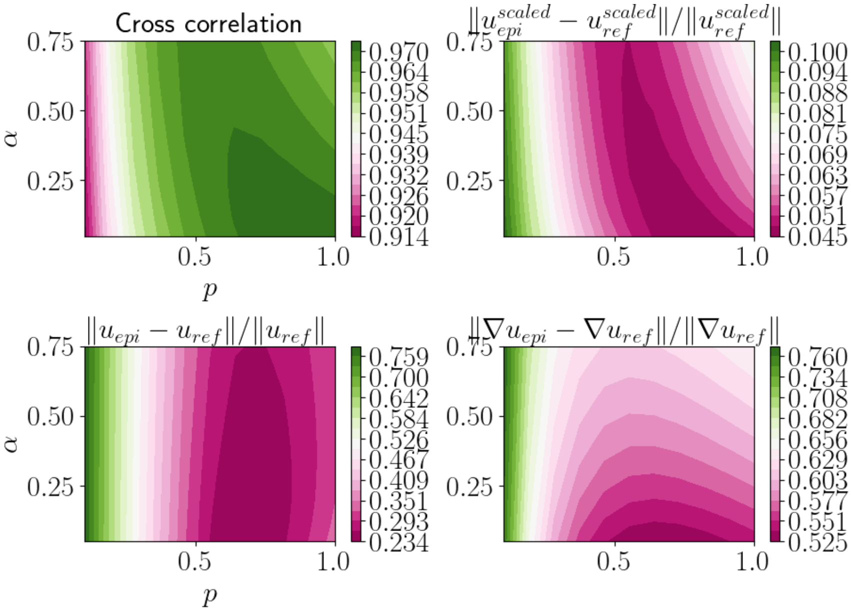
Activation maps

Post-processing of reconstructed temporal u and v


Delay1
Fib1
Fib3
Double1
True AT
Dst V L1
Dst V0
Dst Upp
1.0e+00
50
1.2e+02

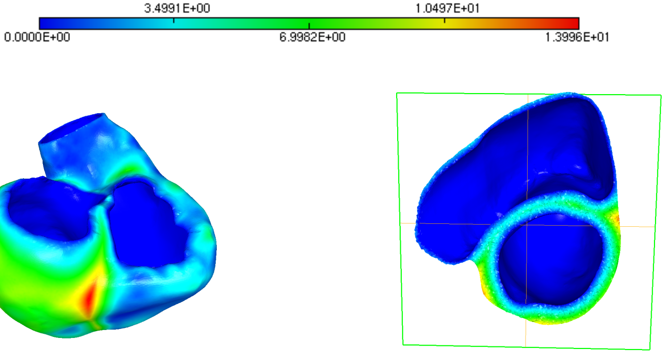
Take home message

- A novel ”Epicardial” model for the inverse problem of cardiac electrophysiology
- Allows to manipulate a model analogous to the bidomain model, vary regularization and functionals
- Still a very difficult inverse problem


Cross correlation
$\|u_{epi}^{scaled} - u_{ref}^{scaled}\| / \|u_{ref}^{scaled}\|$
$\|u_{epi} - u_{ref}\| / \|u_{ref}\|$
$\|\nabla u_{epi} - \nabla u_{ref}\| / \|\nabla u_{ref}\|$
α
p
0.75
0.50
0.25
0.5
1.0
0.970 0.964 0.958 0.951 0.945 0.939 0.932 0.926 0.920 0.914
0.100 0.094 0.088 0.081 0.075 0.069 0.063 0.057 0.051 0.045
0.759 0.700 0.642 0.584 0.526 0.467 0.409 0.351 0.293 0.234
0.760 0.734 0.708 0.682 0.656 0.629 0.603 0.577 0.551 0.525

Focus on h implementation

Crane, Keenan and Weischedel, Clarisse and Wardetzky, Max *The heat method for distance computation*, Communications of the ACM, 2017



Meshes

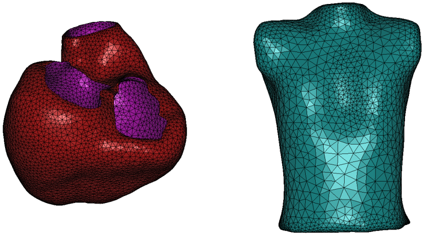


Figure: Forward mesh (data)

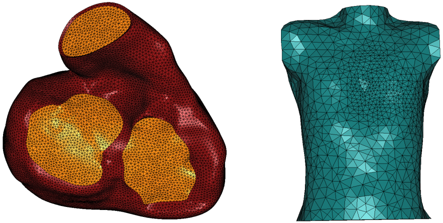


Figure: Inverse mesh

Focus on TV constrained optimization problem

Algorithm 1 Fix point iteration method

Init: $((\bar{u}^0, u^0), v^0)$ Tikhonov solution

while not converged **do**

Update:

$$((\bar{u}^{k+1}, u^{k+1}), v^{(k+1)}) := \operatorname*{arg\,min}_{((\bar{u},u),v)\in\mathcal{E}} \frac{1}{2}\int_{\Gamma_T} \frac{(u - z_T)^2}{\sqrt{(u^k - z_T)^2 + \beta}} + \frac{\varepsilon}{2}\int_{\Gamma_H} \frac{\nabla v \cdot \nabla v}{\sqrt{||\nabla v^k||_2^2 + \beta}} + \frac{\varepsilon_{\text{inv}}}{2}\int_{\Gamma_H} v^2$$

end while

Output: Solution $((\bar{u}^{k+1}, u^{k+1}), v^{(k+1)})$

Curtis R Vogel et Mary E Oman. *Iterative methods for total variation denoising*, SIAM Journal on Scientific Computing (1996).