A continuous-time optimal control approach based on Pontryagin's Maximum Principle with chance constraints

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June 3, 2025

Summary

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Introduction

Introduction

- Continuous-time optimal control with chance constraints to handle uncertainty
- Chance constraints guarantee a threshold of performance α of satisfaction
- We can derive a deterministic equivalent formulation in SOCP (second-order conic programming) [Prékopa, 2013]
- Comparison:
  - Solving the optimal control problem as a nonlinear optimisation problem with integral cost function [Valli et al., 2024]
  - Solving an ODE system as a two-point boundary value problem
- Application to reference trajectory planning generation for autonomous vehicles

Writing the optimal control problem

Problem setup

- Let $n, m \in \mathbb{N}$ be dimensions of state and control vectors respectively $z(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$. Let $t_f \in \mathbb{R}$, $t_f \geq 0$ the final time.
- Integrand $l : \mathbb{R}^n \times \mathbb{R}^m \longmapsto \mathbb{R}$, Integral cost function $J : \mathbb{R}^m \rightarrow \mathbb{R}$, Controls $\mathbf{u} = (u(t))_{t \in [0, t_f]}$ and states $\mathbf{z} = (z(t))_{t \in [0, t_f]}$

Cost function

$$J(\mathbf{u}) = \int_0^{t_f} l(z(t), u(t))\, dt \tag{1}$$

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Optimisation problem

- Let $\alpha \in [0, 1]$
- Control-state equation: $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$,
- Random vector $a \sim \mathcal{N}(\mu, \Sigma)$, with $\mu \in \mathbb{R}^n$ the mean vector and $\Sigma \in \mathbb{R}^{n \times n}$ the covariance matrix, $b \in \mathbb{R}$

Optimisation problem with chance constraint

$$\begin{aligned} \min_{\mathbf{u}} \quad & J(\mathbf{u}) \\ \text{s.t.} \quad & \frac{dz(t)}{dt} = f(z(t), u(t)) \\ & \mathbb{P}(a^T z(t) \leq b) \geq \alpha \end{aligned} \tag{2}$$

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Chance constraint reformulation

- Thanks to [Prékopa, 2013], the chance constraint is equivalent to a second-order conic programming (SOCP) constraint

Chance constraint

$$\mathbb{P}(a^T z(t) \leq b) \geq \alpha \Longleftrightarrow \mu^T z(t) + F^{-1}(\alpha)||\Sigma^{1/2} z(t)||^2 \leq b \qquad (3)$$

with

- $F(\cdot)$ the cumulative distribution function (CDF) of the standard normal distribution $\mathcal{N}(0, 1)$.
- $|| \cdot ||$ the Euclidean norm

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Writing the Hamiltonian

Path constrained optimisation problems for nonlinear dynamic systems

- From [Bryson, 2018], chapter 3.11, let

Constraint on function of the state variable

$$S(z(t)) := \mu^T z(t) + F^{-1}(\alpha) z(t)^T \Sigma z(t) - b \qquad (4)$$

Inequality constraint on function of the state variables only $S(z(t)) \leq 0$

Let

- $\lambda(t) \in \mathbb{R}^n$ the costates associated to $z(t)$
- $\eta(t) \in \mathbb{R}^n$, $\eta(t) \geq 0$ the time-dependent Lagrangian multipliers associated to $S(z(t))$

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Time-dependent Lagrangian multipliers

- Time-dependent Lagrangian multipliers $\eta(t)$ are defined such as

Time-dependent Lagrangian multipliers

$$\begin{cases} \eta(t) = 0 & \text{if } S(z(t)) < 0 \\ \eta(t) > 0 & \text{if } S(z(t)) = 0 \end{cases} \tag{5}$$

Remark

Depending on the problem studied, discontinuities may appear at constraint saturation, leading to singular arcs [Bryson, 2018]

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q-th order state variable inequality constraint

- To control the system at constraint saturation $S(z(t)) = 0$, we need to find $q \in \mathbb{N}$ as the $q - th$ time derivative of the constraint $S(z(t))$ depends explicitly on the command $u(t)$.
- The order q is fixed such as it exists $g : \mathbb{R}^n \times \mathbb{R}^m \longmapsto \mathbb{R}^n$ verifying the following condition:

q-th order state variable inequality constraint

$$\frac{d^{(q)} S(z(t))}{dt^{(q)}} = g(z(t), u(t)) \tag{6}$$

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Tangency conditions

- The following condition must be checked for the command $u(t)$ to be optimal

Tangency conditions

$$\frac{d^{(q)} S(z(t))}{dt^{(q)}} = 0 \quad \text{if } S(z(t)) = 0 \tag{7}$$

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Hamiltonian formulation

- The Hamiltonian writes as

Hamiltonian

$$H(z(t),u(t),\lambda(t),\eta(t)) = l(z(t),u(t)) + \lambda^T(t)\cdot f(z(t),u(t)) + \eta(t)^T \cdot \frac{d^q S(z(t))}{dt^q} \tag{8}$$

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Problem formulation

Euler-Lagrange equations

- The Euler-Lagrange equations give ordinary differential equations to determine the costates $\lambda(t)$

Euler-Lagrange equations for costates $\lambda(t)$

$$\frac{d\lambda(t)}{dt} = -\frac{\partial H(z(t), u(t), \lambda(t), \eta(t))}{\partial z(t)} \tag{9}$$

Transversality conditions and optimal command

- In our problem, there is no constraint at final time, therefore

$$\lambda(t_f) = 0 \tag{10}$$

- The optimal command $u^*(t)$ is obtained by

Optimal command $u^*(t)$

$$u^*(t) = \text{argmin}_{u_{\min} \leq u(t) \leq u_{\max}} H\Big(z(t), u(t), \lambda(t), \eta(t)\Big) \tag{11}$$

- It verifies the condition

$$\frac{\partial H(z(t), u^*(t), \lambda(t), \eta(t))}{\partial u(t)} = 0 \tag{12}$$

Two-Point Boundary Value Problem (TPBVP)

- Finally, we obtain the two-point boundary value problem such as

TPBVP schema

$$\xrightarrow{z(0)}\ \xrightarrow[\lambda(0)?]{}\ \boxed{\begin{cases} \dfrac{dz(t)}{dt} = f(z(t), u(t)) \\ \dfrac{d\lambda(t)}{dt} = -\dfrac{\partial H(z(t), u(t), \lambda(t), \eta(t))}{\partial z(t)} \end{cases}}\ \xrightarrow{z(t_f)}\ \xrightarrow[\lambda(t_f)=0]{} \tag{13}$$

$\lambda^*(0)$ and shooting method

- Therefore, the control is determined by optimal initial values

Optimal initial values $\lambda^*(0)$

$$\lambda^*(0) = \text{argmin}\, ||\lambda(t_f)|| \tag{14}$$

- Classical approach to solve this problem is to use the shooting method [Morrison et al., 1962]
- In our application, we use Levenberg-Marquardt algorithm [Gavin, 2019] with an estimation $\tilde{\lambda}(0)$ as starting point, obtained by reversing the system (13) using results of our previous work [Valli et al., 2024] with costates $\lambda(t_f) = 0$. We use GEKKO optimisation solver [Beal et al., 2018].

Application to trajectory planning problem

- We represent the vehicle controlled (*the ego vehicle*) by its Cartesian coordinates.

Command $u(t)$ and state $z(t)$

$$u(t) = \begin{pmatrix} j_t \\ \omega_t \end{pmatrix} \qquad (15) \qquad\qquad z(t) = \begin{pmatrix} x_t \\ y_t \\ \theta_t \\ v_t \\ a_t \end{pmatrix} \qquad (16)$$

- j_t is the jerk, ω_t the angular velocity
- (x_t, y_t) the Cartesian coordinates, θ_t the heading angle of the ego vehicle, v_t the linear speed and a_t the linear acceleration

Trajectory planning II

- The control-state relationship is given by :

Control-state equation

$$\frac{dz(t)}{dt} = f(z(t), u(t)) = \begin{pmatrix} v_t \cos(\theta_t) \\ v_t \sin(\theta_t) \\ \omega_t \\ a_t \\ j_t \end{pmatrix} \tag{17}$$

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Objective function

- The waypoints $(x_t^{wp}, y_t^{wp}, \theta_t^{wp})_{t\geq 0}$ corresponds to the centre lane of the road, v_r the recommended linear speed

Objective function

$$\int_0^{t_f} \mathbf{w'}_x(t)\left(\frac{x_t}{x_t^{wp}}-1\right)^2+\mathbf{w'}_y(t)\left(\frac{y_t}{y_t^{wp}}-1\right)^2+\mathbf{w'}_h(t)\left(\frac{\theta_t}{\theta_t^{wp}}-1\right)^2 \quad (18)$$

$$+\mathbf{w'}_v\left(\frac{v_t}{v_r}-1\right)^2+\mathbf{w'}_a\cdot a_t^2+\mathbf{w'}_\omega\cdot\omega_t^2+\mathbf{w'}_j\cdot j_t^2$$

$$+\mathbf{w}_p\cdot P(x_t^{tgt}, y_t^{tgt}, x_t, y_t)dt$$

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Adaptive Cruise Control (ACC) feature

- The ACC feature [Liu et al., 2017] is modelled by

Adaptive Cruise Control

$$P(x_t^{tgt}, y_t^{tgt}, x_t, y_t) = \tag{19}$$
$$e^{-\left(\frac{(x_t^{tgt}-x_t)^2+(y_t^{tgt}-y_t)^2}{2}\right)}\left(1+\operatorname{erf}\left(\frac{\operatorname{sign}(x_t^{tgt}-x_t)\sqrt{(x_t^{tgt}-x_t)^2+(y_t^{tgt}-y_t)^2}}{\sqrt{2}}\right)\right)$$

With

$$\forall x \in \mathbb{R} \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{20}$$

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Chance constraints on target distance

- Uncertainty on *target vehicle* position:

$$x_t^{tgt} \sim \mathcal{N}(\mu_{x_t}, \sigma_{x_t}), y_t^{tgt} \sim \mathcal{N}(\mu_{y_t}, \sigma_{y_t}) \tag{21}$$

Equivalent constraints

From [Valli et al., 2024]

$$\mathbb{P}(|x_t - x_t^{tgt} + y_t - y_t^{tgt}| \geq d_{min}) \geq \alpha \iff \tag{22}$$

$$x_t + y_t \leq \mu_{x_t} + \mu_{y_t} + d_{min} + \sqrt{\sigma_{x_t}^2 + \sigma_{y_t}^2} F_N^{-1}(\alpha/2) \tag{23}$$

$$x_t + y_t \geq \mu_{x_t} + \mu_{y_t} - d_{min} + \sqrt{\sigma_{x_t}^2 + \sigma_{y_t}^2} F_N^{-1}(1 - \alpha/2) \tag{24}$$

Experiments setup

Parameter	Function	Value
v_r	Reference linear speed	$12\ m.s^{-1}$
d_{min}	Minimum distance between vehicles	$5\ m$
v_{max}	Maximum linear speed	$40\ m.s^{-1}$
ω_{max}	Maximum angular speed	$\frac{\pi}{6}\ s^{-1}$
a_{max}	Maximum acceleration	$2\ m.s^{-2}$
j_{max}	Maximum jerk	$0.6\ m.s^{-3}$

Table: Parameters' values for urban driving scenarios during the simulation.

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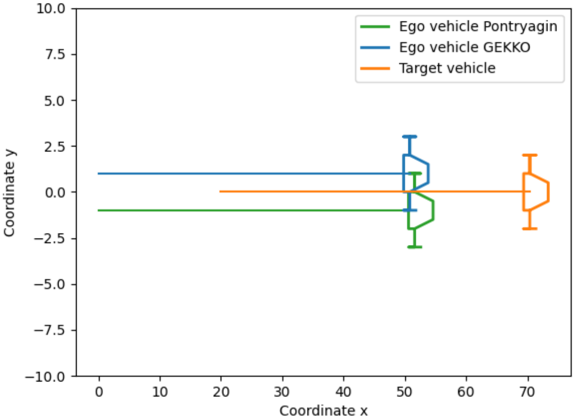
Resolution steps

Algorithm Solving the continuous-time optimal control problem

1: **Initialize** $z(0)$ initial state, t_f final time
2: **Solve** the optimal control problem using GEKKO solver starting from initial state $z(0)$
3: **return** $\left(z^{GEKKO}(t), u^{GEKKO}(t)\right)_{[\![0,t_f]\!]}$
4: **Reverse integration** of the TPBVP starting from $\left(z^{GEKKO}(t_f), \lambda(t_f) = 0\right)$
5: **return** $\left(\tilde{z}(0), \tilde{\lambda}(0)\right)$
6: **Optimise** the initial costates by Levenberg–Marquardt algorithm applied on the TPBVP starting from $\left(z(0), \tilde{\lambda}(0)\right)$
7: **return** $\lambda^*(0)$
8: **Forward integration** of the TPBVP
9: **return** $\left(z^{Pontryagin}(t), u^{Pontryagin}(t)\right)_{[\![0,t_f]\!]}$

Trajectory comparison



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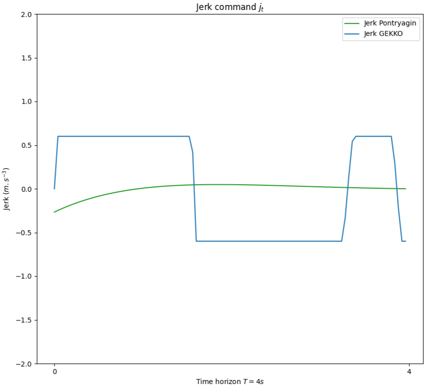
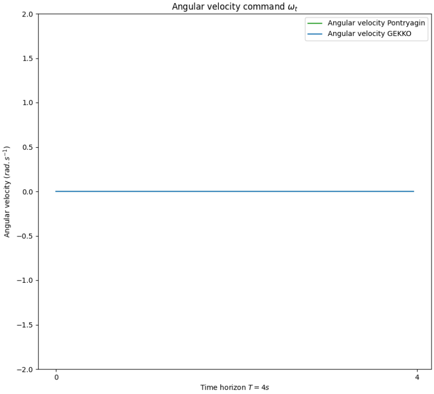
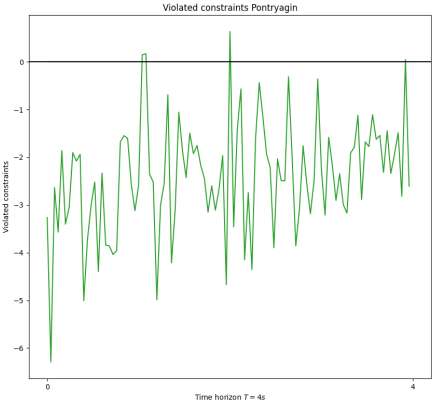
Optimal Commands

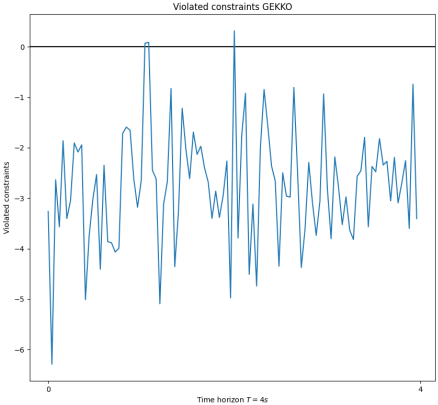




Constraint violations

- Quantity $d_{min} - |x_t^{tgt} - x_t + y_t^{tgt} - y_t|$





Conclusion

- Pontryagin solver achieves smoother command and tighter constraint handling than direct methods used by GEKKO solver
- Better minimise the cost function, but finding initial conditions can be complex
- Further perspectives: study the impact of approximations on the method (derivative of $\mathrm{sign}(\cdot)$, choice of ODE solver, choice of optimisation technique); extend to higher time horizons; apply the approach to other optimal control problems

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Thank you !

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Annexes

- Transversality conditions on the costates $\lambda(t_f)$ depend on the constraints at final time t_f.
- Let $\Psi : \mathbb{R}^n \longmapsto \mathbb{R}^n$ represents the equality constraints on the final state such as $\Psi(z(t_f)) = 0$ and $\nu \in \mathbb{R}^n$ the Lagrange multipliers associated.
- Let $\Phi(t_f) \in \mathbb{R}$ the terminal cost function such that

$$J(\mathbf{u}) = \int_0^{t_f} l(z(t), u(t))\, dt + \Phi(t_f) \tag{25}$$

Transversality conditions

$$\lambda(t_f) = \frac{\partial}{\partial z(t_f)} (\Phi(t_f) + \nu^T \Psi(t_f)) \tag{26}$$

- Time horizon: $T = 4\,s$, time step $\Delta t = 0.04\,s$
- Driving along the road, straight line ($\forall t \in \mathbb{R}^+ \quad y_t = 0, \theta_t = 0$)

GEKKO Solving

$$z(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 13.038 \\ 0 \end{pmatrix} \quad (27) \qquad z^{GEKKO}(t_f) = \begin{pmatrix} 50.81 \\ 0 \\ 0 \\ 12.61 \\ 0.01 \end{pmatrix} \quad (28)$$

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Annex II : Numerical simulation II

Reverse integration

$$\tilde{z}(0) = \begin{pmatrix} -1.662 \\ 0 \\ 0 \\ 14.840 \\ -1.631 \end{pmatrix} \tag{29}$$

$$\tilde{\lambda}(0) = \begin{pmatrix} -4.546 \\ 0 \\ 0 \\ -8.242 \\ -16.277 \end{pmatrix} \tag{30}$$

Annex II : Numerical simulation III

Levenberg–Marquardt optimisation

$$\lambda^*(0) = \begin{pmatrix} 0.572 \\ -9.194 * 10^{-17} \\ 1.460 * 10^{-16} \\ 0.959 \\ 0.533 \end{pmatrix} \tag{31}$$

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Annex II : Optimal cost

- Numerical integration using Simpson's method

Optimal cost for GEKKO solver

$$J^*_{GEKKO} : \int_0^{t_f} l(z^{GEKKO}(t), u^{GEKKO}(t))dt = 8.90 \tag{32}$$

Optimal cost for Pontryagin solver

$$J^*_{Pontryagin} : \int_0^{t_f} l(z^{Pontryagin}(t), u^{Pontryagin}(t))dt = 4.91 \tag{33}$$

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